

Interrelation of creep and relaxation: a modeling approach for ligaments

by

R. S. Lakes[§] and R. Vanderby[¶]
University of Wisconsin, Madison
[§]Department of Engineering Physics
[§]Rheology Research Center
147 Engineering Research Building
1500 Engineering Drive
Madison, WI 53706-1687

^{§¶}Biomedical Engineering Program and
[¶]Division of Orthopedic Surgery
Madison, WI 53792-3228

J Biomech. Engineering, 121, 612-615, Dec. (1999).

Key Words: Ligament, viscoelasticity, creep, relaxation.

Abstract

Experimental data (Thornton, et al., 1997) show that relaxation proceeds more rapidly (a greater slope on a log log scale) than creep in ligament, a fact not explained by linear viscoelasticity. An interrelation between creep and relaxation is therefore developed for ligaments based on a single-integral nonlinear superposition model. This interrelation differs from the convolution relation obtained by Laplace transforms for linear materials. We demonstrate via continuum concepts of nonlinear viscoelasticity that such a difference in rate between creep and relaxation phenomenologically occurs when the nonlinearity is of a strain-stiffening type, i.e. the stress-strain curve is concave up as observed in ligament. We also show that it is inconsistent to assume a Fung-type constitutive law (Fung, 1972) for both creep and relaxation. Using published data of Thornton, et al., (1997) the nonlinear interrelation developed herein predicts creep behavior from relaxation data well ($R = 0.998$). Although data are limited and the causal mechanisms associated with viscoelastic tissue behavior are complex, continuum concepts demonstrated here appear capable of interrelating creep and relaxation with fidelity.

1. Introduction

The most common phenomenological model of the viscoelastic behavior of ligaments is the quasi-linear viscoelasticity model of Fung (1972), which allows time-dependence (viscoelasticity) and strain-dependence (nonlinearity) of the stiffness. This model has been useful in describing a number of experiments with ligaments and tendons (e.g. Haut and Little, 1972; Woo, 1982; Woo et al., 1981; Kwan et al., 1993). Other models include a single integral finite strain formulation (Johnson et al., 1996) and a hyperelastic finite strain formulation (Pioletti et al., 1998).- In all of these studies, experiments were performed with deformation control and, hence, the viscoelastic formulation was only used to describe relaxation response of the tissue. Force driven behavior was not considered. In a more recent study by Thornton et al. (1997), both creep and relaxation were investigated. Their data showed that relaxation proceeds much more rapidly than creep (a greater slope on a log log scale), and they showed that linear viscoelastic theory was not able to phenomenologically model both behaviors with linearly related constitutive coefficients. This rate difference between creep and relaxation was consistent with the more clinically focused experiments of Graf et al. (1994).

Many constitutive relations have been developed for the description of nonlinearly viscoelastic materials (Schapery, 1969, Lockett, 1972; Findley et al., 1976). Single-integral formulations are relatively simple but can only handle a subset of viscoelastic phenomena. Multiple-integral formulations (Green and Rivlin, 1957) are more versatile but also more complicated. The interrelation of creep and relaxation is well known for linearly viscoelastic materials. The interrelation has been studied for some kinds of nonlinear response such as irreversible creep of metals (Arutyunyan, 1966; Popov, 1947). These studies are not suitable for ligament. Since they do not involve superposition they cannot be adapted to the general load histories which occur in the body. The interrelation has been studied for materials -describable by a particular multiple integral formulation in polymers (Lai and Findley, 1968) or involving single exponentials (Molinari, 1973). These formulations have not been used to describe ligaments, but offer that potential. The objective of this study is therefore to develop an interrelation between creep and relaxation for ligament based on a single-integral quasi-linear nonlinear superposition viscoelastic model. This model is then used to interpret data existing in the literature (Thornton et al., 1997).

2. Analysis

2.1 Analysis of linearly viscoelastic materials

In linearly viscoelastic materials it is straightforward to develop a relationship between creep compliance $J(t)$ as it depends on time t , and relaxation response $E(t)$. $E(t)$ is an axial (Young's) relaxation modulus as distinguished from a shear modulus $G(t)$. The derivative theorem and convolution theorem for the Laplace transform are used to convert constitutive equations which are the following Boltzmann integrals,

$$\epsilon(t) = \int_0^t E(t - \tau) \frac{d\epsilon(\tau)}{d\tau} d\tau, \quad \sigma(t) = \int_0^t J(t - \tau) \frac{d\sigma(\tau)}{d\tau} d\tau. \quad (1)$$

to

$\epsilon(s) = s E(s) \epsilon(s)$ and $\sigma(s) = s J(s) \sigma(s)$ respectively. Here s is the transform variable.

In the integral forms, we consider the time scale to begin just prior to time zero, in order that step function load histories beginning at zero may be accommodated without difficulty with the delta function which arises from the derivative.

So

$$\frac{\epsilon(s)}{\sigma(s)} = s E(s) \text{ and } \frac{\sigma(s)}{\epsilon(s)} = \frac{1}{sJ(s)}. \quad (2)$$

Setting these stress strain ratios equal,

$$E(s)J(s) = \frac{1}{s^2}. \quad (3)$$

Taking the inverse transform, using the convolution theorem and the relation

$$\mathcal{L}[t] = 1/s^2,$$

$$\int_0^t J(t - \tau) E(\tau) d\tau = \int_0^t E(t - \tau) J(\tau) d\tau = t. \quad (4)$$

The relationship is implicit. Explicit relationships can be developed via Laplace transformation provided a specific analytical form is given for $E(t)$ or $J(t)$. Power law behavior in time is particularly simple:

$$E(t) = At^{-n}. \quad (5)$$

Taking the Laplace transform of $E(t)$, and using Eq. (3), and recognizing $\Gamma(n)$ as the gamma function, defined as follows for $n > 0$,

$$\Gamma(n) = \int_0^\infty t^{n-1} e^{-t} dt. \quad (6)$$

$$J(t) = \frac{1}{A \Gamma(n) \Gamma(n+1)} t^n. \quad (7)$$

$$E(t) = \frac{\sin n}{n} \frac{1}{J(t)}. \quad (8)$$

So for this example the creep function is also a power law in time.

2.2 Analysis of nonlinearly viscoelastic materials

The following simple nonlinear relation allows for prediction of history dependence. This single-integral form is called nonlinear superposition, which allows the relaxation function to depend on strain level.

$$\sigma(t) = \int_0^t E(t - \tau, \epsilon(\tau)) \frac{d\epsilon(\tau)}{d\tau} d\tau. \quad (9)$$

A similar equation may be written in the compliance formulation. Some care is required if step strain is applied, since the delta function which arises from the derivative should multiply a function which is continuous at the delta (Schapery, 1999). One may express $E(t, \epsilon)$ as a product, or more generally as a sum of products or else allow a nonzero risetime in relaxation, to avoid

mathematical difficulty. In the formulation of Fung (1972), the strain-dependent modulus is *separable* into the product of a function of time and a function of strain:

$$E(t, \epsilon) = E_t(t) g(\epsilon). \quad (10)$$

Separable nonlinear superposition has been widely used in the modeling of ligaments and other tissue.

Nonlinearly viscoelastic materials cannot be analyzed via Laplace transforms. The interrelation between creep and relaxation is developed directly here. Write the time-dependent strain due to a constant stress σ_c as a sum of immediate and delayed Heaviside step functions in time $H(t)$,

$$\epsilon(t) = \epsilon(0)H(t) + \sum_{i=0}^N \epsilon_i H(t - t_i). \quad (11)$$

Each step strain in the summation gives rise to a relaxing component of stress in view of the definition of the relaxation function. Here we assume there is no effect from interactions between the step components, hence we consider single-integral type nonlinear response and exclude response which must be describable by a multiple integral formulation.

$$\sigma_c = \sigma(0)E(t, \epsilon) + \sum_{i=0}^N \epsilon_i E(t - t_i, \epsilon). \quad (12)$$

Divide by σ_c and use the definition of the creep compliance,

$$1 = J(0)E(t, \epsilon) + \sum_{i=0}^N J_i E(t - t_i, \epsilon_i). \quad (13)$$

Pass to the limit of infinitely many fine step components to obtain a Stieltjes integral, with ϵ as a time variable of integration.

$$1 = J(0)E(t, \epsilon) + \int_0^t E(t - \epsilon, \epsilon) \frac{dJ(\epsilon, \epsilon)}{d\epsilon} d\epsilon. \quad (14)$$

The relationship is implicit and for the linear case it is equivalent to Eq. 4. To develop an explicit form, assume the creep behavior to be separable into a stress-dependent portion and a power law in time.

$$J(t, \epsilon) = A(\epsilon) t^n = \{g_1 + g_2 \epsilon + g_3 \epsilon^2 + \dots\} t^n. \quad (15)$$

Assume the relaxation behavior to be as follows. A separable form does not give rise to a solution.

$$E(t, \epsilon) = \{f_1 t^{-n} + f_2 (t) t^{-2n} + f_3 (t) t^{-3n} + \dots\}. \quad (16)$$

The creep strain is $\epsilon(t) = J(t) \sigma_c = at^n$, with a as the strain amplitude. Substitute Eq. 15 and 16 into 14, and recognize that $J(0)$ for the power law vanishes,

$$1 = \int_0^t \{f_1 (t-\epsilon)^{-n} + f_2 a (t-\epsilon)^n (t-\epsilon)^{-2n} + f_3 a^2 (t-\epsilon)^{2n} (t-\epsilon)^{-3n} + \dots\} \{g_1 + g_2 \epsilon + g_3 \epsilon^2 + \dots\} n \epsilon^{n-1} d\epsilon. \quad (17)$$

Factor the stress-dependent and time-dependent portions,

$$1 = \{g_1 + g_2 \epsilon + g_3 \epsilon^2 + \dots\} \int_0^t \{f_1 + f_2 a \epsilon + f_3 a^2 \epsilon^2 + \dots\} n (t-\epsilon)^{-n} \epsilon^{n-1} d\epsilon. \quad (18)$$

The integral portion gives results $(\frac{1}{n} \sin n \pi)$ identical to the linear case, Eq. 8, by Laplace transformation of the integral and an identity involving the gamma function. The stress-dependent portion is related to the strain dependent portion by inversion of a power series (Abramowitz and Stegun, 1965).

$$f_1 = 1/g_1. \quad (19)$$

$$f_2 = -g_2/g_1^3. \quad (20)$$

$$f_3 = (2g_2^2 - g_1g_3)/g_1^5. \quad (21)$$

Although the creep behavior was assumed separable, the corresponding relaxation behavior is not separable. Therefore it is inconsistent to assume a separable, Fung-type constitutive law for both creep and relaxation. That is, if one used a Fung-type model for creep and another Fung-type model for relaxation data from the same specimen, the resulting constitutive coefficients would not be interrelated. The nonlinear material exhibits a relaxation response which contains a sum of power law terms, as given in Eq. 16.

2.3 Interpretation of ligament data

The ligament data in Fig. 3 of Thornton et al. (1997), described as 'typical', were scanned from the original article and digitized, and are replotted on a log-log scale in Fig. 1. From about 20 seconds to 1000 seconds, both creep and relaxation curves are fitted by power laws in time, with correlation coefficients $R = 0.998$ or better. Creep goes as $J(t) \propto t^n$ with a slope $n = 0.028$ and relaxation goes as $E(t) \propto t^{-m}$ with $m = 0.0716$. Relaxation predicted from creep using a linear model gives $m = 0.0215$. This slope is close to the same value as creep, as expected from Eq. 8 but differs from the actual slope of the relaxation curve. Creep predicted from relaxation gives $n = 0.0759$, which differs from the actual slope of the creep curve. Predictions based on Laplace transformation (linear viscoelasticity) are consequently poor.

The nonlinear interrelation developed above allows relaxation to proceed faster than creep as a result of the series of power law terms. Stress-strain data of Thornton et al. (1997) were scanned and curve fitted. Below 6% strain, a single term power law curve fit gives $\sigma = 2883 \epsilon^{1.93}$, with a correlation $R = 0.999$, and a two term curve fit with integer exponents gives $\sigma = 3162 \epsilon^2 + 11152 \epsilon^3$, with a correlation $R = 0.999$, and with the quadratic term dominating below 6 % strain. Curve fitting, with a free exponent, of the actual relaxation data yields a relaxation function $E(t) = 1.0946 t^{-0.0716}$ and $R = 0.999$ for a single power term. A single term fit constrained to be quadratic gives $E(t) = 1.002 t^{-0.056}$ with $R = 0.97$. Finally, $E(t) = 0.406 t^{-0.056} + 0.694 t^{-0.084}$ with $R = 0.999$ for a two term fit. Here the exponents are constrained to be two and three times the exponent for creep, as required from the nonlinear conversion. As shown in Fig. 1, relaxation predicted from creep combined with the nonlinearity embodied in the stress-strain curve agrees well with the relaxation actually observed.

The power law behavior occupies less than two decades of the time scale, therefore results over more decades would be helpful in future curve fitting. Even so, a nonlinear interrelation based on a quadratic fit of the scanned stress-strain curve offers superior prediction of relaxation from creep in comparison with a linear interrelation, as shown in Fig. 1. The precision of the stress-strain plot is not known; if a cubic term is admitted, the comparison between observed and predicted relaxation is even better. We remark that Woo (1982) used an exponential function $\sigma = A(e^{B\epsilon} - 1)$ to fit the stress-strain behavior, and we observe that over a restricted range of strain, this is approximated by a quadratic or a cubic function, depending on the range. For larger strains, it is expected that more terms in the power series will be needed for adequate curve fits.

As for times less than 10 seconds, the data of Thornton et al. (1997) Fig. 3 deviate from a power-law. However the rise-time of the transient load for creep and extension for relaxation was not reported. In view of the effect of rise-time, one ordinarily records transient data beginning at a time a factor of ten longer than the rise-time of the applied load or strain; otherwise errors can occur (Turner, 1973). Interpretation of this segment of the time scale is not definitive until we know the rise time. Ligament creep and relaxation are likely follow a power law in time over a wider range than the 1.7 decades considered here. The ligament relaxation results of Woo (1982), covering five decades of time down to 0.6 seconds (the rise-time was 0.25 seconds), can be fitted to a power

law $t^{-0.0625}$ with a correlation coefficient $R = 0.995$. This behavior is similar, for time greater than 20 seconds, to that observed by Thornton et al. (1997).

3. Discussion

The goal of this study was to formulate an interrelationship between creep and relaxation for ligaments based on continuum concepts and upon available data. Since no experimental data were collected with the aim of verifying the interrelationship, verification is limited to data available from Thornton, et al. (1997). The interrelations developed here appear capable of interrelating creep and relaxation with fidelity. Since mechanical behaviors for various ligaments and tendons are different, the above interrelationships must be individually tested for fidelity with these and other tissues.

Ligaments are nonlinearly viscoelastic at physiologic strains of 5 % or less and also at large strain associated with injury. Their behavior is commonly described by a single-integral nonlinear superposition constitutive equation referred to as quasi-linear. The quasi-linearity means that there is no interaction between transients (such as step or pulse functions in time) which occur at different times. For example, recovery following creep occurs at the same rate as creep in a material which obeys nonlinear superposition. However one cannot use Laplace transforms to relate creep and relaxation in a nonlinear superposition type material. The relation developed here for power-law creep predicts creep and relaxation to proceed at substantially different rates. Indeed, different rates can be observed in data of Thornton et al. (1997) Fig. 3 in a medial collateral ligament and from Graf et al. (1994) in a patellar tendon. The present interrelation is restricted to a particular, albeit commonly used, constitutive equation.

Constitutive equations deal with continuum concepts of stress and strain, not with material microstructure. Knowledge of the microstructure allows one to explore the causal mechanisms responsible for material behavior. In tendon and ligament, recruitment of collagen fibers occurs at increasing loads (Viidik, 1972). Fiber recruitment gives rise to a nonlinearity of a strain-stiffening type, i.e. the stress-strain curve is concave up. The present analysis demonstrates that, based on continuum concepts, such nonlinearity causes relaxation to proceed more rapidly than creep. By contrast, in a linear material, power law creep and relaxation curves have the same slope. In a structural vein, after observing differences in creep and relaxation behaviors, Thornton et al. (1997) speculated that the differences were due to progressive recruitment of collagen fibers during creep. If true, then a micromechanical fiber recruitment model (e.g. Hurschler et al., 1997) with a simpler viscoelastic formulation may give added insight into the physical basis and mechanisms of the viscoelastic behavior. We observe as a caveat that ligament viscoelasticity studies have been based on end-to-end (structural) testing which may not fully reveal material properties. Alternatively, the difference in creep and relaxation may arise from different rates of tension driven fluid exudation (Vanderby et al., 1999) and osmotically driven fluid imbibation. Regardless of the source however, the above nonlinear viscoelastic interrelation provides a robust description for both force driven and deformation driven elongations of ligaments based on available data.

Nonlinear viscoelasticity is known in polymers (Lai and Findley, 1968) and in compact bone (Lakes, et al., 1979). In some cases, the nonlinear superposition approach appears adequate, but evidence has been found for time interactions describable by the more general multiple integral constitutive relations (Ward and Onat, 1963; Lakes and Katz, 1979). It is not known if a separable (Fung type) single-integral formulation is adequate to describe ligament because the necessary experiments have not yet been done. One can test for whether the kernel is separable by performing tests at different load levels. One can test for time interactions by applying load histories with multiple steps. Since we have found that for creep behavior assumed separable, the corresponding relaxation behavior is not separable, we recommend a future experimental program involving creep and relaxation studies at different load and strain levels. Such studies will reveal aspects of nonlinear viscoelasticity not revealed by current studies which combine a single creep or relaxation curve and a nonlinear stress-strain curve.

Acknowledgment

The technical comments of Richard A Schapery are gratefully acknowledged.

References

- Abramowitz, M. and Stegun, I., 1965, *Handbook of Mathematical Functions*, Dover, NY.
- Arutyunyan, N. Kh., 1966, *Some problems in the theory of creep*, Pergamon, Oxford, UK.
- Findley, W. N., Lai, J. S., and Onaran, K., 1976. *Creep and Relaxation of Nonlinear Viscoelastic Materials*, North Holland, Amsterdam.
- Fung, Y. C., 1972, "Stress strain history relations of soft tissues in simple elongation", in *Biomechanics, its Foundations and Objectives*, ed. Fung, Y. C., Perrone, N. and Anliker, M., Prentice Hall, Englewood Cliffs, NJ.
- Graf, B. K., Vanderby, R. Jr., Ulm, M. J., Rogalski, R. P., and Thielke, R. J., 1994, "Effect of preconditioning on the viscoelastic response of primate patellar tendon", *Athrosocopy*, Vol. 10, pp. 90-96.
- Green, E. and Rivlin, R. S., 1957, "The mechanics of non-linear materials with memory", *Arch. Rational Mech. Anal.* Vol. 1, pp. 1-21.
- Haut RC, Little RW. (1972): A constitutive equation for collagen fibers. *J Biomech*, 5(5):423--430.
- Hurschler, C., Loitz-Ramage, B., and Vanderby, R., 1997, "A structurally based stress-stretch relationship for tendon and ligament," *ASME Journal of Biomechanical Engineering*, Vol. 119, pp. 392-399.
- Johnson, G. A., Livesay, G. A., Woo, S. L-Y., and Rajagopal, K. R., 1996, "A Single Integral Finite Strain Viscoelastic Model of Ligaments and Tendons," *ASME Journal of Biomechanical Engineering*, Vol. 118, pp. 221-226.
- Kwan, M. K., Lin, T. H-C., and Woo, S. L-Y., 1993, "On the Viscoelastic Properties of the Anteromedial Bundle of the Anterior Cruciate Ligament," *Journal of Biomechanics*, Vol. 26, pp. 447-452.
- Lai, J. S. Y. and Findley, W. N., 1968, "Prediction of uniaxial stress relaxation from creep of nonlinear viscoelastic material", *Trans. Soc. Rheology*, Vol. 12, pp. 243-257.
- Lakes, R. S., Katz, J. L., and Sternstein, S. S., 1979, "Viscoelastic properties of wet cortical bone: Part I, torsional and biaxial studies." *Journal of Biomechanics*, Vol. 12, pp. 657-678.
- Lakes, R. S. and Katz, J. L., 1979, "Viscoelastic properties of wet cortical bone: Part III, A non-linear constitutive equation," *Journal of Biomechanics*, Vol. 12, pp. 689-698.
- Lockett, F. J., 1972, *Nonlinear Viscoelastic Solids*, Academic, NY.
- Molinari, A., 1973, "Sur la relation entre fluage et relaxation en viscoélasticité non linéaire", *Comptes Rendus de l'Academie des Sciences, Serie I, Mathematiques*, Vol. 27, A, pp. 621-623.
- Pioletti DP, Rakotomanana LR, Benvenuti JF, Leyvraz PF. (1998): Viscoelastic constitutive law in large deformations: application to human knee ligaments and tendons. *J Biomech*, 31(8):753-757.
- Popov, E. P., 1947, "Correlation of tension creep tests with relaxation tests", *J. Appl. Mechanics*, Vol. 14, pp. 135-142.
- Schapery, R. A., 1969, "On the characterization of nonlinear viscoelastic materials", *Polymer Engineering and Science*, Vol. 9, pp. 295-310.
- Schapery, R. A., 1999, private communication.
- Thornton, G. M., Oliynyk, A., Frank, C. B., and Shrive, N. G., 1997, "Ligament creep cannot be predicted from stress relaxation at low stress: a biomechanical study of the rabbit medial collateral ligament", *J. Orthop. Research.*, Vol. 15, pp. 652-656.
- Turner, S., 1973, "Creep in glassy polymers", in *The Physics of Glassy Polymers*, Ed. Howard, R. H., J. Wiley, NY.
- Vanderby, R., Thielke, R. J., Grood, E. S., and Graf, B. K., 19998, "Tension induced changes in a rat medial collateral ligament," accepted for *ASME Journal of Biomechanical Engineering*.

Viidik, A., 1972, "Simultaneous mechanical and light microscopic studies of collagen fibers", *Z. Anat. Entwickl. Gesch.* Vol. 136, pp. 204-212.

Ward, I. M. and Onat, E. T., 1963, "Non-linear mechanical behaviour of oriented polypropylene", *J. Mech. Phys. Solids* Vol. 11, pp. 217-229.

Woo, S. L-Y., Gomez, M. A., and Akeson, W. H., 1981, "The Time and History-Dependent Viscoelastic Properties of the Canine Medial Collateral Ligament," *ASME Journal of Biomechanical Engineering*, Vol. 103, pp. 293-298.

Woo, S. L-Y., 1982, "Mechanical Properties of Tendons and Ligaments I. Quasistatic and Nonlinear Viscoelastic Properties," *Biorheology*, Vol. 19, pp. 385-396.

Figure 1

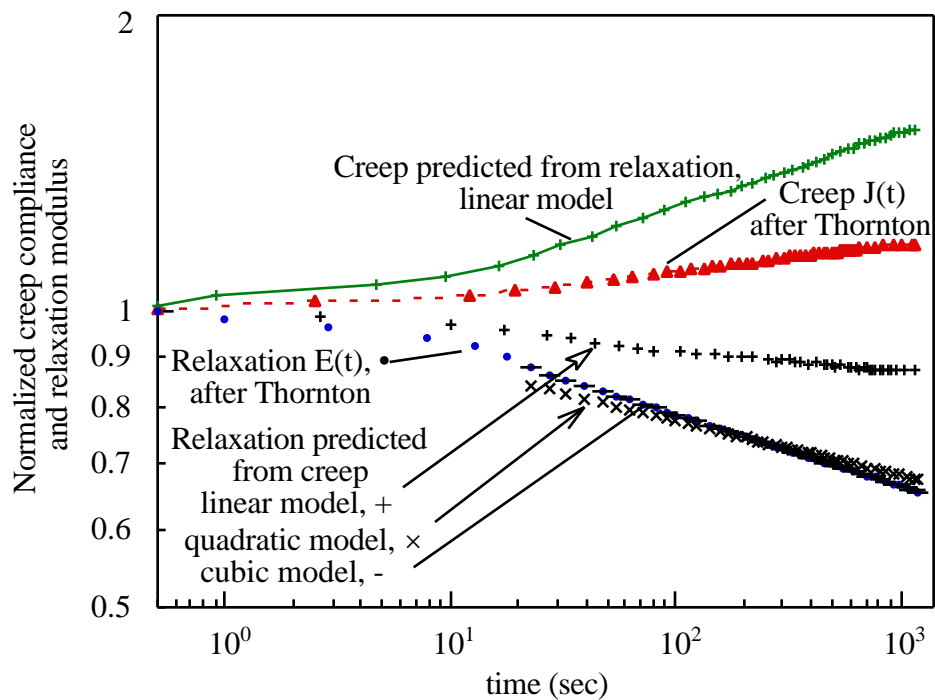


Figure 1. Normalized data of Thornton et al. (1997) for ligament creep $J(t)$, \blacktriangle , and relaxation $E(t)$, \bullet , replotted on a log-log scale. Relaxation predicted from creep using a linearly viscoelastic (Laplace transform) approach, +, does not agree with observed relaxation, \bullet . Use of a nonlinear approach such as a quadratic model, x, or a cubic model, *, offers better predictive power.