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VISCOELASTIC PROPERTIES OF WET CORTICAL BONE—III. A NON-LINEAR CONSTITUTIVE EQUATION*†

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Abstract – A constitutive equation is proposed to characterize the viscoelastic properties of wet human compact bone measured in torsion at body temperature. The equation is developed, using a spectrum of relaxation times,

$$H(\tau) = \begin{bmatrix} \log\tau & \tau_1 \le \tau \le \tau_2 \\ 0 & \tau < \tau_1,\ \tau > \tau_2 \end{bmatrix}$$

called the 'triangle' spectrum. The transient and dynamic viscoelastic moduli are derived from this. Applications of this distribution to other viscoelastic data as well as to dielectric systems, are discussed. Nonlinear effects observed in bone are described using the first two terms of a multiple-integral expansion. The dynamic properties of two types of nonlinear solid are obtained from constitutive equations based on transients, and applications to bone mechanics are discussed.

INTRODUCTION

An enormous amount of research has been done in recent years into the physical properties of human tissues, in particular into their mechanical properties. Prior to a successful theoretical treatment of biomechanical problems, a mathematical description of the relation between stress and strain for the tissue in question must be available. However, many experimenters in the past have not attempted to formulate such descriptions, possibly because their data covered only small segments of the time, frequency and strain domains.

In the first paper of this series (Lakes *et al.*, 1979)‡, measurements of the torsional dynamic and relaxation properties of cortical bone have been reported. These experiments were performed on wet human and bovine bone at body temperature, over wide intervals in time, frequency and strain. The present study has two objectives: the first is to formulate a constitutive equation characterizing these data. In developing this equation, a distribution of relaxation times is obtained, which appears not to have been treated previously. The second objective is to ascertain what are the measureable viscoelastic functions associated with this spectrum.

Presently available data for human bone suggest that, to a good approximation, the time-dependent and strain-dependent aspects of the viscoelastic response may be dealt with separately. Therefore, for simplicity, the material in the following two sections is discussed within the framework of linear viscoelastic theory. Although the language of mechanical relaxation is used, the results are equally applicable to relaxation in linear dielectric, magnetic or piezoelectric systems.

RELAXATION SPECTRA

Phenomena such as mechanical, dielectric and magnetic relaxation are often described by means of a spectrum of relaxation times. In terms of the symbols commonly used for viscoelasticity in shear, the relaxation spectrum $H(\tau)$ is related to the measured viscoelastic functions by the following:

$$G(t) = Ge + \int_{-\infty}^{\infty} H(\tau) e^{-t/\tau} \,\mathrm{d}\log\tau \tag{1}$$

$$G'(\omega) = Ge + \int_{-\infty}^{\infty} H(\tau) \frac{\omega^2\tau^2}{1+\omega^2\tau^2} \,\mathrm{d}\log\tau \tag{2}$$

$$G''(\omega) = \int_{-\infty}^{\infty} H(\tau) \frac{\omega\tau}{1+\omega^2\tau^2} \,\mathrm{d}\log\tau, \tag{3}$$

where $G(t)$ is the relaxation modulus, $G'(\omega)$ is the dynamic storage modulus, $G''(\omega)$ is the dynamic loss modulus and $Ge = \lim_{t\to\infty} G(t)$. These integrals may be inverted with some difficulty to obtain the spectrum

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‡ This will be referred to henceforth as Part 1.

from the measured quantities; however, in practice, approximations are generally used. The role of the spectrum in the structure of viscoelasticity theory is treated by Gross (1953).

For the case in which the spectrum is arbitrarily sharp [i.e. a delta function, $\frac{H(\tau)}{\tau} = \Delta G\,\delta(\tau-\tau_0)$] the following are obtained (the so-called Debye equations):

$$G(t) = Ge + \Delta G \cdot e^{-t/\tau_0} \tag{4}$$

$$G'(\omega) = Ge + \Delta G \frac{\omega^2\tau_0^2}{1+\omega^2\tau_0^2} \tag{5}$$

$$G''(\omega) = \Delta G \frac{\omega\tau_0}{1+\omega^2\tau_0^2}. \tag{6}$$

Such expressions are obtained theoretically for situations in which there is a single characteristic rate at which the system in question readjusts itself to equilibrium, i.e. a single relaxation time. Experimentally measured quantities generally have a more gradual time or frequency dependence than is predicted by the Debye equations.

Various empirical distributions of relaxation times have been compiled by Gross (1953). For example, the 'box distribution'

$$\underset{B[\tau_1,\tau_2]}{H(\tau)} = \begin{bmatrix} 1 & \tau_1 \le \tau \le \tau_2 \\ 0 & \tau > \tau_2,\ \ \tau < \tau_1 \end{bmatrix} \tag{7}$$

has been found to be useful in the description of polymeric solids (Tobolsky, 1950) and soft tissues (Fung, 1972). The measured viscoelastic quantities corresponding to the box spectrum may be expressed in terms of well-known functions:

$$\underset{B[\tau_1,\tau_2]}{G(t)} = Ge + [Ei(-t/\tau_1) - Ei(-t/\tau_2)] \tag{8}$$

$$\underset{B[\tau_1,\tau_2]}{G'(\omega)} = Ge + 1/2 \log\left(\frac{1+\omega^2\tau_2^2}{1+\omega^2\tau_1^2}\right) \tag{9}$$

$$\underset{B[\tau_1,\tau_2]}{G''(\omega)} = (\tan^{-1}\omega\tau_2 - \tan^{-1}\omega\tau_1), \tag{10}$$

where $Ei(x)$ is the well-known and extensively tabulated exponential–integral function (see e.g. Lowan, 1940) and the subscript $B[\tau_1, \tau_2]$ denotes the two time parameters in the box spectrum. A plot of equations (7)–(10) is given in Fig. 1, assuming that $[\tau_1, \tau_2] = [10^{-2}, 10^2]$. Now, provided that $\tau_2 \gg \tau_1$, the box distribution gives rise to a loss G'' which is essentially constant in the middle of the domain, while $G'(\omega)$ and $G(t)$ decrease linearly with $\log 1/\omega$ and $\log \tau$, respectively. For $\tau_2 \gtrsim \tau_1$, the measurable functions corresponding to different spectra do not differ dramatically.

A 'wedge' distribution (which is wedge-shaped in a log–log plot)

$$H(\tau) = \begin{bmatrix} \tau^{-1/2} & \tau_1 \le \tau \le \tau_2 \\ 0 & \tau > \tau_2,\ \ \tau < \tau_1 \end{bmatrix} \tag{11}$$

was introduced and analyzed by Tobolsky (1960) for the description of the glass–rubber transition of polyisobutylene.

The lognormal distribution

$$H(\tau) = \frac{b}{\sqrt{\pi}} e^{-b^2z^2}; \qquad z = \log(\tau/\tau_m) \tag{12}$$

was originally introduced by Weichert (1893) and developed further by Nowick and Berry (1961). Here τ_m is the time at which the spectrum peaks, and $1/b$ defines its width. Nowick and Berry assert that this spectrum has a theoretical rather than a purely empirical basis, since a given relaxation mechanism is likely to operate in a range of atomic environments distributed in a Gaussian fashion about some mean value. While there may be some justification for this, it should be noted that there are many situations in which an asymmetric loss peak (i.e. G'' vs $\log\omega$) is observed; such cases cannot be treated using the lognormal spectrum. The dynamic and static moduli corresponding to this distribution cannot be expressed as a finite combination of well-known functions; however, Nowick and Berry provide tables obtained by numerical methods.

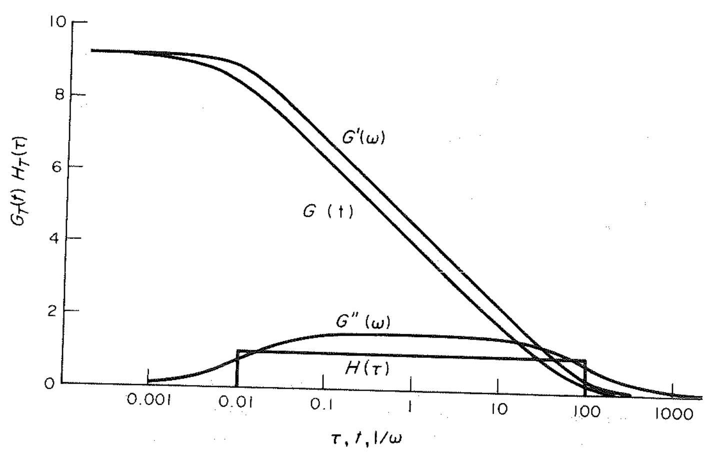

Fig. 1. Relaxation and dynamic moduli corresponding to the box spectrum $\underset{B[10^{-2},10^2]}{H(\tau)}$. Arbitrary units of stress on ordinate. Arbitrary units of time on abscissa.

THE TRIANGLE SPECTRUM

Motivation

Experimental measurement of the stress–relaxation behavior of human compact bone in torsion gives rise to the results which cannot be readily modeled using only previously published spectra such as those described above. The slope of the relaxation curve is seen to increase in magnitude with the logarithm of time. The relaxation curve, further, is much broader than would be expected if the spectrum contained a single relaxation time. This suggests that the corresponding relaxation spectrum increases with $\log \tau$ also. Therefore, the dynamic and relaxation functions corresponding to the 'triangle spectrum'

$$\underset{T[\tau_1,\tau_2]}{H(\tau)} \equiv \begin{bmatrix} \log \tau & \tau_1 < \tau < \tau_2 \\ 0 & \tau > \tau_2, \quad \tau < \tau_1 \end{bmatrix} \tag{13}$$

are germane to this treatment and are developed in Appendix A.

APPLICATION TO BONE

A number of relaxation spectra and their associated linear viscoelastic response functions have been examined. In order to formulate a constitutive equation, the objective is to find a spectrum such that the corresponding $G(t)$, $G'(\omega)$ and $G''(\omega)$ fit the experimental results. A first step in doing this is to obtain an approximate spectrum for the region of 'small' strain using the approximations

$$H(\tau) \cong -\left.\frac{\mathrm{d}G(t)}{\mathrm{d}\log t}\right|_{t=\tau}; \qquad H(\tau) \cong \frac{2}{\pi} G''(\omega)\bigg|_{\omega=1/\tau}$$

(Ferry, 1970). The resulting curve is then fitted with empirical spectra for which the associated measured functions are known, and the parameters varied until a close approximation to the observed data can be obtained. The spectrum thus obtained is

$$\frac{H(\tau)}{G_{std}} = 0.00318\underset{T[1,10^6]}{H(\tau)} + 0.006\underset{B[10^{-5},10^2]}{H(\tau)} + 0.002\underset{B[10^{-5},10^{-3}]}{H(\tau)} + 0.004\delta(\tau-0.2)\tau, \tag{14}$$

where the times are in seconds and $G_{std} = 0.590 \times 10^6$ lb/in.^{2} = 4.068 GN/M^2. The corresponding relaxation function is

$$\frac{G_0(t)}{G_{std}} = 0.00318\underset{T[1,10^6]}{G(t)} + 0.006\underset{B[10^{-5},10^2]}{G(t)} + 0.002\underset{B[10^{-5},10^{-3}]}{G(t)} + 0.004\,\mathrm{e}^{-t/0.2} + 0.69. \tag{15}$$

The third term in each of these expressions contributes to the loss modulus above 100 Hz. While this region was not studied in the present experiments, other data obtained using dry canine radial bone (Thompson, 1971) indicated that the torsional loss tangent in the domain 370–2500 Hz lies between 0.016 and 0.019. Although wet human bone would not necessarily behave in the same way, these results must suffice as a first approximation until more data are available. An increase in the loss tangent for human bone above 100 Hz is also suggested by the results of low-temperature experiments upon human bone below 100 Hz (described earlier in Lakes *et al.*, 1979). Since bone is thermorheologically complex, a stronger statement than this cannot be justified.

Now the nonlinearities observed in the present study of the torsional response of bone are primarily of a strain-dependent type. Such effects can be described by an equation of nonlinear superposition which is therefore proposed to describe this response:

$$\sigma(t) = \int_{-\infty}^{t} \{G[t-\tau, \varepsilon(\tau)] - Ge\}\frac{\mathrm{d}\varepsilon}{\mathrm{d}\tau}\mathrm{d}\tau + Ge \cdot \varepsilon. \tag{16}$$

An equation similar to this has been used to describe the behavior of soft tissue by Fung (1972) and is also of use in describing synthetic polymers. Since the relaxation curves obtained for human bone in the present experiments are parallel within experimental error, the kernel may be separated

$$G(t', \varepsilon) = G_0(t') \cdot A(\varepsilon). \tag{17}$$

For the present data

$$A(\varepsilon) = a_1 - a_2\,\mathrm{e}^{-(a_3\varepsilon)^2}, \tag{18}$$

where $a_1 = 1.055$, $a_2 = 0.07$, $a_3 = 550$. The kernel in the constitutive equation (16) becomes

$$G(t,\varepsilon) = \{0.00318\underset{T[1,10^6]}{G(t)} + 0.006\underset{B[10^{-5},10^2]}{G(t)} + 0.002\underset{B[10^{-5},10^{-3}]}{G(t)} + 0.004\,\mathrm{e}^{-t/0.2} + 0.692\}[a_1 - a_2\,\mathrm{e}^{-(a_3\varepsilon)^2}]G_{std} \tag{19}$$

Equations (16) and (19) are supported by experimental data in the domains $10^{-3} \le t \le 10^5$ sec, $3.4 \times 10^{-5} \le \varepsilon \le 1.7 \times 10^{-3}$. While it is likely that these equations are valid for smaller strains than the above minimum, a breakdown of the formulation is to be expected at larger strains, since no provision is made for yield or fracture behavior.

Nonlinear equations more general than equation (16) have been proposed to account for non-superposable behavior observed in polymers. For example, Green and Rivlin (1957) have obtained a multiple-integral series expansion relating the stress to the history of the displacement gradients. In one dimension, for small strains, this may be written (Ward and Onat, 1963)

$$\sigma(t) = \int_{-\infty}^{t} G_0(t-\tau)\frac{\mathrm{d}\varepsilon}{\mathrm{d}\tau}\mathrm{d}\tau + \int_{-\infty}^{t}\int_{-\infty}^{t} G_1(t-\tau_1, t-\tau_2) \times \frac{\mathrm{d}\varepsilon}{\mathrm{d}\tau_1}\frac{\mathrm{d}\varepsilon}{\mathrm{d}\tau_2}\mathrm{d}\tau_1\mathrm{d}\tau_2 + \cdots. \tag{20}$$

Although this expression is general in the sense that arbitrary nonlinear effects are describable by it, it has proven to be unwieldy, since the determination of the kernel functions G necessary to describe even 'simple' nonlinear behavior involves an enormous number of experiments. Pipkin and Rogers (1968) have devised an alternative series representation in which the first kernel is completely determined by a set of single-step experiments, the second kernel by two-step experiments, etc. This series may be written

$$\sigma(t) = \int_{-\infty}^{t} \frac{\partial R[\varepsilon(\tau), t-\tau]}{\partial \varepsilon} \frac{d\varepsilon}{d\tau} d\tau + \frac{1}{2} \int_{-\infty}^{t} \int_{-\infty}^{t} \times \frac{\partial R_1[\varepsilon(\tau_1), t-\tau_1]}{\partial \varepsilon} \frac{\partial R_2[\varepsilon(\tau_2), t-\tau_2]}{\partial \varepsilon} \times \frac{d\varepsilon}{d\tau_1} \frac{d\varepsilon}{d\tau_2} d\tau_1 d\tau_2 + \cdots + \frac{1}{n!} \int \cdots \int \times \frac{\partial R_1[\varepsilon(\tau_1), t-\tau]}{\partial \varepsilon} \cdots \frac{\partial Rn[\varepsilon(\tau_n), t-\tau_n]}{\partial \varepsilon} \times \frac{d\varepsilon}{d\tau_1} \cdots \frac{d\varepsilon}{d\tau_n} d\tau_1 \ldots d\tau_n + \cdots \quad (21)$$

The kernels may also be expressed as follows

$$\frac{\partial R_0}{\partial \varepsilon} \equiv G_0; \qquad \frac{\partial R_1}{\partial \varepsilon} \frac{\partial R_2}{\partial \varepsilon} \equiv G_1, \quad \text{etc.}$$

This formulation has the advantage that a truncation containing the first few terms generally approximates the behavior of real materials better than a corresponding truncation of the Green–Rivlin series. The first term of equation (21) is equivalent to the nonlinear superposition equation (16), while the first term of equation (20) is simply the Boltzmann superposition integral.

To detect deviations from nonlinear superposition in the behavior of a material, transient experiments using a single step-function are inadequate; more complex strain histories must be used. For example, a 'relaxation and recovery' experiment in which a pulse strain history

$$\varepsilon(t) = \begin{bmatrix} \varepsilon_0 & 0 < t < t_1 \\ 0 & t < 0, \quad t > t_1 \end{bmatrix} \quad (22)$$

is applied to a material obeying equation (21), yields the following stress response for $t > t_1$:

$$\sigma(t) = [G_0(t, \varepsilon_0) - G_0(t-t_1, \varepsilon_0)]\varepsilon_0 - [G_1(t-t_1, \varepsilon_0; t_0, \varepsilon_0) + G_1(t, \varepsilon_0; t-t_1, \varepsilon_0) - G_1(t-t_1, \varepsilon_0; t-t_1, \varepsilon_0) - G_1(t, \varepsilon_0; t, \varepsilon_0)] \times \frac{\varepsilon_0^2}{2} - \text{higher order terms.} \quad (23)$$

Determination of the higher order kernels is possible using multi-step strain-histories; generally, a large number of experiments is necessary. The recovery behavior for human bone as measured in Part I deviates from what would be expected if nonlinear superposition were valid, by less than 2%. These deviations occurred at the largest strains and longest times used in the experiments. One possible form for the kernel G_1 which would describe this is

$$G_1[t-\tau_1, \varepsilon(\tau_1), t-\tau_2, \varepsilon(\tau_2)] = 0.37 \frac{[\log(t-\tau_1+1)\log(t-\tau_2+1)]}{e^{t-2(\tau_1+\tau_2)}+1} \times \tanh|\tau_2 - \tau_1|. \quad (24)$$

No experimental evidence was seen that bone is not a fading-memory material at the strains used. Kinder and Sternstein (1976) have shown that for a fading-memory material, the higher-order terms in equation (21) generate only transient stress responses, when the strain history consists of a single step function, i.e. the strain is constant. Therefore, the above expression has been constructed so as to be asymptotic to zero in time. Data which could be used to ascertain the kernels G_2 and higher were not taken.

The preceding has been confined to a one-dimensional treatment of the torsional behavior of compact bone. In a three-dimensional analysis, the anisotropy of bone must be taken into account. The simplest constitutive equation which includes anisotropy is that for an anisotropic elastic solid:

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}. \quad (25)$$

Using ultrasonic techniques, the elements of the modulus tensor C_{ijkl} have been determined by Lang (1969) for bovine bone and by Yoon and Katz (1976) for human bone (see Table 1). For a linearly viscoelastic anisotropic solid, the appropriate equation is

$$\sigma_{ij}(t) = \int_{-\infty}^{t} C_{ijkl}(t-\tau) \frac{d\varepsilon_{kl}}{d\tau} d\tau, \quad (26)$$

while for solids obeying nonlinear superposition,

$$\sigma_{ij}(t) = \int_{-\infty}^{t} C_{ijkl}(t-\tau, \varepsilon_{kl}) \frac{d\varepsilon_{kl}}{d\tau} d\tau. \quad (27)$$

Observe that the kernel obtained for bone in equation (20) corresponds to the tensor element $C_{2323}(t, \varepsilon_{23})$.

Equations (26) and (27) contain no coupling between the rate at which one element of stress σ_{ij} relaxes and those elements of strain ε_{kl} for which C_{ijkl} is zero. Evidence of coupled relaxation kinetics has been found, however, in a number of polymers. Sternstein and Ho (1972) have presented an extension of linear

Table 1. Elements of the elastic modulus tensor for compact bone (units in GN/m²)

Elastic constant: Reduced notation	Elastic constant: Full notation	Yoon and Katz (1976) Dried human femur	Lang (1969) Fresh bovine phalanx
C_{11}	C_{1111}	23.4	19.7
C_{33}	C_{3333}	32.5	33.4
C_{44}	C_{2323}	8.71	8.20
C_{12}	C_{1122}	9.06	10.2
C_{13}	C_{1133}	9.11	11.2
C_{66}	C_{1212}	7.17	3.80

viscoelasticity theory to describe such behavior in isotropic solids. For an anisotropic, nonlinear solid with kinetic coupling, equation (27) may be rewritten:

$$\sigma_{ij}(t) = \int_{-\infty}^{t} C_{ijkl}[t-\tau, \varepsilon_{kl}, \Phi(\varepsilon_{mn})] \frac{d\varepsilon_{kl}}{d\tau} d\tau, \quad (28)$$

where the function Φ must be chosen in such a way as to preserve the material's symmetry, if any. For example, Φ can depend only upon the three strain invariants for an isotropic solid.

In the biaxial experiments upon bone described in Part I, the torsional viscoelastic behavior of bone was observed in the presence of an axial tensile stress. The appropriate special case of the above equation must therefore be written implicitly:

$$\sigma_{23}(t) = \int_{-\infty}^{t} C_{2323}(t-\tau, \varepsilon_{23}, \sigma_{33}) \frac{d\varepsilon_{23}}{d\tau} d\tau. \quad (29)$$

The effect of a superposed tensile stress is relatively small; the kernel G_0 in equation (19) may be modified as follows to describe the increase in the loss tangent at high frequencies:

$$\frac{G(t,\varepsilon)}{G_{std}} = \Big[0.00318 \underset{T[1,10^6]}{G(t)} + 0.006 \underset{B[10^{-5},10^2]}{G(t)} + 0.002 \underset{B[10^{-5},10^{-3}]}{G(t)} + 0.004\, e^{-t/0.2} + 0.0013 \underset{B[10^{-4},10^{-2}]}{G(t)} \cdot \frac{\sigma_{33}}{17.2} + 0.692 \Big] \cdot A(\varepsilon) \cdot \quad (30)$$

The slowing of the torsional recovery by an axial stress is described by the following correction to the kernel G_1 appearing in equation (24):

$$G_1 = 0.37 \frac{[\log(t-\tau_1+1)\log(t-\tau_2+1)]^2}{e^{t-2(\tau_1+\tau_2)}+1} \times \tanh|\tau_2-\tau_1| \left[1 + \frac{\sigma_{33}}{17.2}\right], \quad (31)$$

where σ_{33} is in units of MN/m^2.

DYNAMIC BEHAVIOR OF NONLINEAR SOLIDS

It is of interest to examine the response to sinusoidal excitation of nonlinear solids which obey different types of constitutive equations. For linear materials, an analysis of the material response to oscillatory strain or stress histories leads to the introduction of the complex moduli or compliances (see e.g. Gross, 1953; Ferry, 1970). The stress response of solids obeying equation (16) (nonlinear superposition) and solids obeying equation (20) (Green–Rivlin series) to dynamic strain histories is treated in the following analysis.

Consider first (Lakes, 1975) a solid describable by the nonlinear superposition integral 16 in which the kernel $G(t,\varepsilon)$ is separable: $G(t,\varepsilon) = G_0(t) + A(t)B(\varepsilon)$. If the strain is given by $\varepsilon(\tau) = \varepsilon_0 \sin \omega\tau$, then equation (16) becomes

$$\sigma(t)/\varepsilon_0 = G'(\omega)\sin \omega t + G''(\omega)\cos \omega t + A'(\omega)B(\varepsilon_0 \sin \omega t) + A''(\omega)B(\varepsilon_0 \cos \omega t), \quad (32)$$

where

$$G'(\omega) = \omega \int_0^\infty [G_0(t') - Ge]\sin \omega t'\, dt' + Ge$$

$$G''(\omega) = \omega \int_0^\infty [G_0(t') - Ge]\cos \omega t'\, dt'$$

$$A'(\omega) = \omega \int_0^\infty A(t')\sin \omega t'\, dt'$$

$$A''(\omega) = \omega \int_0^\infty A(t')\cos \omega t'\, dt'.$$

Specific forms for equation (32) can be obtained by expanding B as is detailed in Appendix B.

Now the response of the solid to a strain history containing a single frequency component, $\varepsilon(t) = a_1 \sin \omega_1 t$, can be obtained as a special case of the development in Appendix B by letting $a = a_1, \omega = \omega_1$. The single frequency strain is then seen to generate harmonics (integer multiples of the original frequency ω_1) in the stress. It is shown that an analysis of these harmonics can enable one to calculate $\hat{G}_n(\omega, \omega \ldots \omega)$; however, a test involving a single frequency component in applied strain is inadequate to obtain $G_n(\omega_1 \ldots \omega_n)$. Multi-frequency histories must be used to obtain these quantities; from these, the original kernel functions in equation (20) may be extracted by means of a suitable transformation procedure.

The results of these analyses may be summarized as follows.

(1) For both the 'Green–Rivlin solid' and a solid obeying nonlinear superposition, the stress response to a sinusoidally varying strain contains the original frequency, harmonics at integer multiples of this frequency, and a constant or 'D.C.' stress.

(2) If the response $\sigma(t)/\varepsilon_0$ is invariant to the sign of ε_0, as must be the case in shear for an isotropic solid or in torsion about the symmetry axis of an axisymmetric solid, all terms in the representation of $\sigma(t)$ containing odd powers of ε_0 vanish, so that all even harmonics as well as the static 'D.C.' stress also vanish.

(3) The response in tension/compression is not restricted in this way and may contain all harmonics plus a constant stress.

(4) In musical language, these statements may be expressed as follows: vibrated in shear with strain history equivalent to the tone of the flute, the solid responds with a stress corresponding to the tone of the clarinet, while in compression it could respond with the tone of the saxophone.

(5) The response of the two types of solid is indistinguishable if single frequency excitation is used. A strain excitation containing several frequencies will generate 'interactions' in the Green–Rivlin solid; however, the stress response of the solid obeying nonlinear superposition will be equivalent to the sum of the responses to the individual frequencies; no

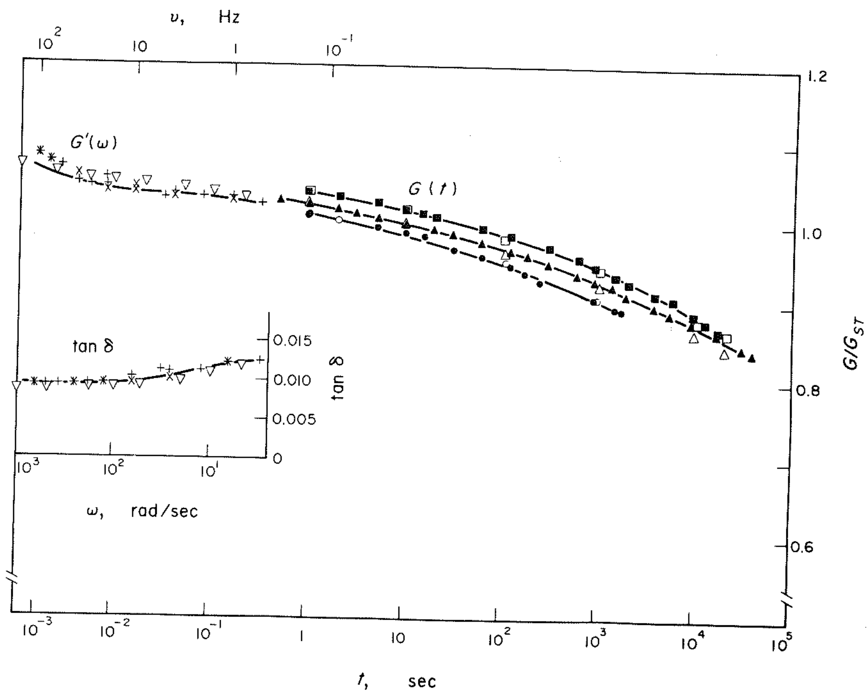

Fig. 2. Viscoelastic response of wet human tibial bone No. 5 at body temperature: experimental and modelled behavior. $G_{std} = 4.068$ GN/m^2.

Experimental data	Modelled behaviour		
●	○	$\gamma = 8.5 \times 10^{-5}$	relaxation
▲	△	$\gamma = 8.5 \times 10^{-4}$	relaxation
■	□	$\gamma = 1.7 \times 10^{-3}$	relaxation
+	▽	$\gamma = 3.4 \times 10^{-5}$	dynamic
×	▽	$\gamma = 3.4 \times 10^{-4}$	dynamic

interactions occur in this case.

(6) The dynamic response at the driving frequency $G'(\omega)$ can be less strain-dependent (for identical maximum strain levels and for $t = 1/\omega$) than the relaxation response $G(t)$.

This last phenomenon has been observed in bone and is described earlier in Lakes *et al.* (1979). The constitutive equation developed in the present study is capable of correctly modeling this effect, as shown in Fig. 2. Harmonic generation in bone is also described earlier (Lakes *et al.*, 1979); this effect is relatively weak at the strains used in the experiments. Excessive harmonic generation in bone *in vivo*, were it to occur, could create problems resulting from the deleterious effect of high frequency vibration on the articular cartilage (Radin *et al.*, 1973).

APPLICATION TO OTHER SYSTEMS

Materials other than bone appear to behave in a fashion consistent with the triangle distribution of relaxation times developed in Appendix A. For example, Tobolsky (1960) reports a relaxation curve for polyisobutylene which he fits with a box spectrum (Fig. 3). Clearly, in the six decades of time-scale to the

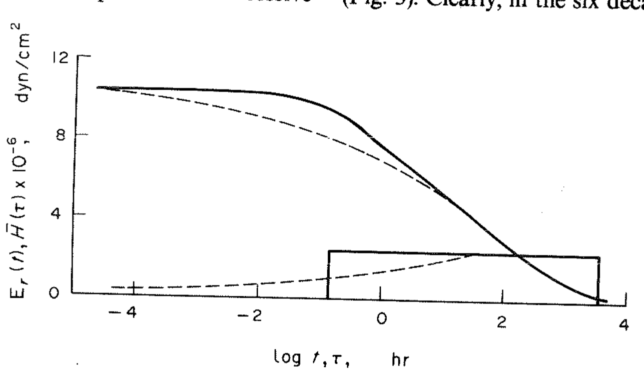

Fig. 3. Young's relaxation modulus $E_r(t)$ for polyisobutylene (after Tobolsky). Dotted line: experimental modulus and calculated spectrum $\bar{H}(\tau)$. Solid line: assumed box spectrum and calculated modulus.

left, the behavior will be approximated much more closely by a triangle spectrum. Other examples, including dielectric relaxation data, may be found in the literature.

DISCUSSION

A constitutive equation has been developed to describe the torsional behavior of wet human compact bone at body temperature. The relaxation spectrum used for this purpose describes the behavior of a number of polymers as well. Although this equation models the experimental data fairly accurately, it has some limitations. First, it can be expected to be valid only in a finite domain of time-scale and strain-level; specifically, it contains no information regarding yield and fracture. Second, the kernel, equation (19), represents only one element of the viscoelastic modulus tensor, which, for high frequencies, contains five independent elements. Finally, the nonlinear, non-superposable response of bone to complex straining histories, is not included. Although many investigations of the viscoelastic and ultimate properties of bone have been previously reported, a complete characterization of the behavior of wet bone at body temperature is not available. The experiments necessary to obtain this remain a subject for future investigations.

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REFERENCES

Ferry, J. D. (1970) *Viscoelastic Properties of Polymers.* Wiley, New York.

Fung, Y. C. (1972) Stress–strain history relations of soft tissues in simple elongation. *Biomechanics, Its Foundations and Objectives* (Edited by Fung. Y. C., Perrone, N. and Anliker, M.). Prentice-Hall, Englewood Cliffs.

Green, E. and Rivlin, R. S. (1957) The mechanics of non-linear materials with memory. *Archs ration. Mech. Analysis* **1**, 1–21.

Gross, B. (1953) *Mathematical Structure of the Theories of Viscoelasticity.* Hermann, Paris.

James, R. C. (1966) *Advanced Calculus.* Wadsworth, Belmont.

Kinder, D. and Sternstein, S. S. (1976) A path-dependent variable approach to non-linear viscoelastic behaviour. *Trans. Soc. Rheol.* **20**, 119–140.

Kourganoff, V. (1963) *Basic Methods in Transfer Problems.* Dover, New York.

Lakes, R. S. (1975) Ph.D. Dissertation, Department of Physics and Astronomy, Rensselaer Polytechnic Institute, New York.

Lakes, R. S., Katz, J. L. and Sternstein, S. S. (1979) Viscoelastic properties of wet cortical bone—I. Torsional and biaxial studies. *J. Biomechanics* **12**, 657–678.

Lang, S. B. (1969) Elastic coefficients of animal bone. *Science, N.Y.* **165**, 281.

Lockett, F. J. and Gurtin, M. E. (1964) Frequency response of nonlinear viscoelastic solids. Brown University Technical Report, NONR 562(10), NONR 562(30).

Lowan, A. N. (1940) *Tables of Sine, Cosine and Exponential Integrals.* National Bureau of Standards Computation Lab.

Nowick, A. S. and Berry, B. S. (1961) Lognormal distribution function for describing anelastic and other relaxation processes, I. Theory and numerical computations. *I.B.M. J. Res. Dev.* **5**, 297–312.

Pipkin, A. C. and Rogers, T. G. (1968) A non-linear integral representation for viscoelastic behaviour. *J. Mech. Phys. Solids* **18**, 59–72.

Radin, E. L., Parker, M. G., Pugh, J. W., Steinberg, R. S., Paul, I. L. and Rose, R. M. (1973) Response of joints to impact loading—III. Relationship between trabecular micro-fractures and cartilage degeneration. *J. Biomechanics* **6**, 51–57.

Sternstein, S. S. and Ho, T. C. (1972) Biaxial stress relaxation in glassy polymers: Polymethylmethacrylate. *J. appl. Phys.* **43**, 4370–4383.

Thompson, G. (1971) Experimental studies of lateral and torsional vibration of intact dog radii. Ph.D. Thesis, Biomedical Engineering, Stanford University.

Tobolsky, A. B. (1960) *Properties and Structure of Polymers.* Wiley, New York.

Ward, I. M. and Onat, E. T. (1963) Non-linear mechanical behaviour of oriented polypropylene. *J. Mech. Phys. Solids* **11**, 217–229.

Weichert, A. (1893) Gestze der elastischen Nachwirkung für constante Temperatur. *Analn Phys.* **50**, 335–348.

Yoon, H. S. and Katz, J. L. (1976) Ultrasonic wave propagation in human cortical bone – II. Measurements of elastic properties and microhardness. *J. Biomechanics* **9**, 459–464.

APPENDIX A

Derivation of the Functions G(t), G′(ω), G″(ω)

The relationship between the spectrum $H(\tau)$ and the relaxation modulus $G(t)$ given in equation (1) is equivalent to

$$G(t) = Ge + \int_0^\infty \frac{H(\tau)}{\tau}\, e^{-t/\tau}\, d\tau. \qquad \text{(A-1)}$$

If $H(\tau)$ is the triangle spectrum given by equation (13), then

$$G(t) = \int_{\tau_1}^{\tau_2} \frac{\log \tau}{\tau}\, e^{-t/\tau}\, d\tau + Ge. \qquad \text{(A-2)}$$

Letting $y = t/\tau$,

$$G(t) = \int_{y_1}^{y_2} \frac{e^{-y}}{y} \log\frac{y}{t}\, dy + Ge. \qquad \text{(A-3)}$$

Integrating by parts,

$$G(t) = \log\frac{y}{t}\, Ei(-y)\Big|_{y_1}^{y_2} - \int_{y_1}^{y_2} \frac{Ei(-y)}{y}\, dy + Ge$$

$$= \log\frac{y}{t}\, Ei(-y)\Big|_{y_1}^{y_2} + \int_{y_1}^{y_2} \frac{E_1(y)}{y}\, dy + Ge, \qquad \text{(A-4)}$$

where $-Ei(-y) = \int_y^\infty e^{-x}/x \,.\, dx$ is the well known exponential-integral function and $E_1(y) \equiv -Ei(-y)[y > 0]$. Now equation (A-4) may be rewritten as follows:

$$G(t) = \log\frac{y}{t}\, Ei(-y)\Big|_{y_1}^{y_2} + \int_{y_1}^{\infty} \frac{E_1(y)}{y}\, dy - \int_{y_2}^{\infty} \frac{E_1(y)}{y}\, dy + Ge. \qquad \text{(A-5)}$$

These integrals cannot be reduced to a finite combination of elementary functions. However, Kourganoff (1963) has evaluated them in connection with an astrophysical problem. He defines the function $E_1^{(2)}(y)$ as follows:

$$E_1^{(2)}(y) = \int_y^\infty \frac{E_1(x)}{x}\,dx$$

and proves that

$$E_1^{(2)}(y) = \tfrac{1}{2}(\log y + \gamma)^2 + \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{(-1)^n y^n}{n^2 n!}, \quad \text{(A-6)}$$

where γ is Euler's constant ($\gamma \approx 0.5572\ldots$).
Recalling that $y = t/\tau$, equation (A-5) becomes

$$\underset{T[\tau_1,\tau_1]}{G(t)} = \log\frac{1}{\tau_2} Ei\left(\frac{t}{\tau_2}\right) - \log\frac{1}{\tau_1} Ei\left(\frac{t}{\tau_1}\right) + E_1^{(2)}\left(\frac{t}{\tau_1}\right) - E_1^{(2)}\left(\frac{t}{\tau_2}\right) + Ge. \quad \text{(A-7)}$$

The dynamic storage modulus $G'(\omega)$ is given in terms of the spectrum $H(\tau)$ by

$$G'(\omega) = \int_0^\infty \frac{H(\tau)}{\tau}\frac{\omega^2\tau^2}{1+\omega^2\tau^2}\,d\tau + Ge. \quad \text{(A-8)}$$

For $H(\tau)$ equal to the triangle spectrum 13, and, for the present, dropping Ge

$$G'(\omega) = \int_{\tau_1}^{\tau_2} \tau\log\tau \frac{\omega^2}{1+\omega^2\tau^2}\,d\tau. \quad \text{(A-9)}$$

Integrating by parts and observing that

$$\int \tan^{-1}\omega\tau\,d\tau = \tau\tan^{-1}\omega\tau - \frac{1}{2\omega}\log\left(\frac{1}{\omega^2}+\tau^2\right),$$

$$G'(\omega) = \left[\omega\tau\tan^{-1}\omega\tau(\log\tau - 1) + \tfrac{1}{2}\log\left(\frac{1}{\omega}+\tau^2\right)\right]\Bigg|_{\tau_1}^{\tau_2} - \omega I(\omega,\tau)\Bigg|_{\tau_1}^{\tau_2}, \quad \text{(A-10)}$$

where $I(\omega,\tau) \equiv \omega\int\tan^{-1}\omega\tau\log\tau\,d\tau$. This integral cannot be expressed as a finite combination of elementary functions. Expanding the arctangent

$$\tan^{-1}\omega\tau = \sum_{k=0}^{\infty} (-1)^k \frac{(\omega\tau)^{2k+1}}{2k+1} \quad [\omega\tau \le 1] \quad \text{(A-11)}$$

$$\tan^{-1}\omega\tau = \frac{\pi}{2} - \sum_{k=0}^{\infty}(-1)^k \frac{1}{(2k+1)(\omega\tau)^{2k+1}} \quad [\omega\tau \ge 1], \quad \text{(A-12)}$$

the following are obtained

$$I_> = \int\left[\frac{\pi}{2}\sum_{k=0}^{\infty}(-1)^k\frac{1}{(2k+1)(\omega\tau)^{2k+1}}\right]\log\tau\,d\tau \quad [\omega\tau \ge 1] \quad \text{(A-13)}$$

$$I_< = \int\left[\sum_{k=0}^{\infty}(-1)^k\frac{(\omega\tau)^{2k+1}}{2k+1}\right]\log\tau\,d\tau. \quad [\omega\tau \le 1] \quad \text{(A-14)}$$

Now with the bounded convergence theorem (James, 1966) the summations and integrations may be interchanged

$$I_> = \left\{\frac{\pi}{2}(\tau\log\tau - \tau) - \sum_{k=0}^{\infty}(-1)^k \times \frac{1}{2k+1}\int\log\tau(\omega\tau)^{-(2k+1)}\,d\tau\right\}. \quad \text{(A-15)}$$

But

$$\int(\omega\tau)^n\log\tau\,d\tau = \omega^n\tau^{n+1}\left[\frac{\log\tau}{n+1} - \frac{1}{(n+1)^2}\right];$$

$$n \ne 0;$$

$$\int\frac{\log\tau}{\omega\tau}\,d\tau = \frac{(\log\tau)^2}{2\omega}. \quad \text{(A-16)}$$

So

$$I_> = \frac{\pi}{2}\omega\tau(\log\tau - 1) - \frac{(\log\tau)^2}{2} + \sum_{k=1}^{\infty}\frac{(-1)^k(\omega\tau)^{-2k}}{2k+1}\left[\frac{\log\tau}{2k} + \frac{1}{4k^2}\right], \quad [\omega\tau \ge 1]. \quad \text{(A-17)}$$

Similarly,

$$I_< = \sum_{k=0}^{\infty}(-1)^k\frac{(\omega\tau)^{2k+2}}{2k+1}\left[\frac{\log\tau}{2k+2} - \frac{1}{(2k+2)^2}\right] \quad [\omega\tau \le 1]. \quad \text{(A-18)}$$

Now if ω is such that $\omega\tau_1 < 1$ and $\omega\tau_2 > 1$,

$$\underset{T[\tau_1,\tau_2]}{G'(\omega)} = \left[\omega\tau\tan^{-1}\omega\tau(\log\tau - 1) + \tfrac{1}{2}\log\left(\frac{1}{\omega^2}+\tau^2\right)\right]\Bigg|_{\tau_1}^{\tau_2} - \left\{I_>(\omega,\tau)\Bigg|_{\tau=1/\omega}^{\tau_2} + I_<(\omega,\tau)\Bigg|_{\tau=\tau_1}^{1/\omega}\right\}. \quad \text{(A-19)}$$

If, however, $\omega\tau_2 < 1$, the expression in the { } brackets becomes $\{I_<(\omega,\tau)|_{\tau=\tau_1}^{\tau_2}\}$, while if $\omega\tau_2 > 1$, the corresponding expression is $\{I_>(\omega,\tau)|_{\tau=\tau_1}^{\tau_2}\}$.

Now considering the dynamic loss modulus $G''(\omega)$ in terms of the spectrum $H(\tau)$, equation (3) becomes

$$G''(\omega) = \int_0^\infty \frac{H(\tau)}{\tau}\frac{\omega\tau}{1+\omega^2\tau^2}\,d\tau. \quad \text{(A-20)}$$

Again, letting $H(\tau)$ be the triangle spectrum and integrating by parts,

$$\underset{T[\tau_1,\tau_2]}{G'(\omega)} = [\log\tau\tan^{-1}\omega\tau]\Big|_{\tau=\tau_1}^{\tau_2} - [K(\omega,\tau)]\Big|_{\tau=\tau_1}^{\tau_2}, \quad \text{(A-21)}$$

where

$$K(\omega,\tau) \equiv \omega\int_{\tau_1}^{\tau_2}\frac{\tan^{-1}\omega\tau}{\omega\tau}\,d\tau. \quad \text{(A-22)}$$

Recalling equations (A-11) and (A-12) and proceeding in the same manner as for G',

$$K_> = \frac{\pi}{2}\log\omega\tau + \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2(\omega\tau)^{2k+1}} \quad [\omega\tau \ge 1] \quad \text{(A-23)}$$

$$K_< = \sum_{k=0}^{\infty}\frac{(-1)^k(\omega\tau)^{2k+1}}{(2k+1)^2} \quad [\omega\tau \le 1]. \quad \text{(A-24)}$$

No difficulty is encountered in integrating through $\omega\tau = 1$ in this case, so

$$\underset{T[\tau_1,\tau_2]}{G''(\omega)} = [\log\tau\tan^{-1}\omega\tau]\Big|_{\tau=\tau_1}^{\tau_2} - [K(\omega)]\Big|_{\tau=\tau_1}^{\tau_2}. \quad \text{(A-25)}$$

The relaxation and dynamic function $\underset{T[1,10^3]}{H(\tau)}$ and $\underset{T[1,10^6]}{H(\tau)}$ obtained using these relations from the triangle spectra and plotted in Figs. 4 and 5, respectively.

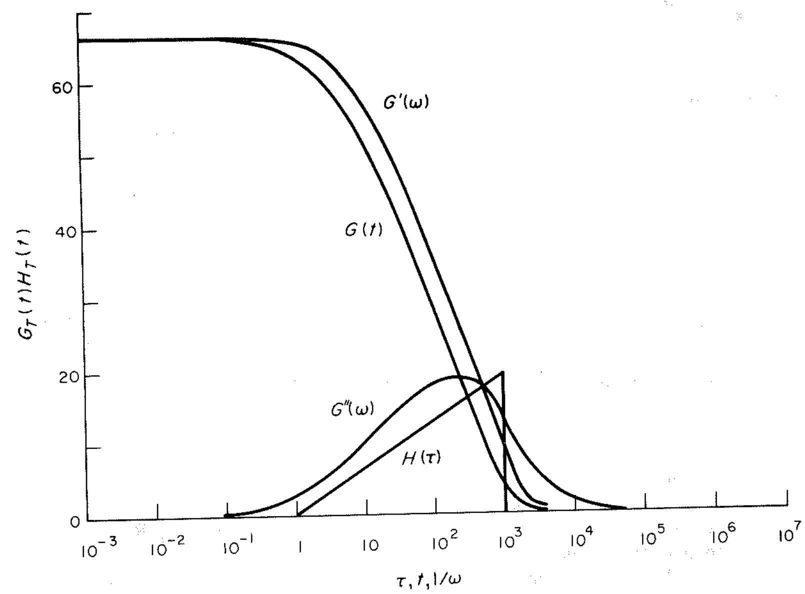

Fig. 4. Relaxation and dynamic moduli corresponding to the triangle spectrum $H(\tau)_{T[1,10^3]}$. Arbitrary units of stress on ordinate. Arbitrary units of time on abscissa.

APPENDIX B

Development of a Nonlinear Constitutive Equation in the Frequency Domain

Expanding B in equation (32) as a power series, eliminating powers of the trigonometric functions by means of multiple-angle identities, and collecting terms, the following is obtained:

$$\begin{aligned}\frac{\sigma(t)}{\varepsilon_0} = {} & \sin \omega t \left\{ G'(\omega) + A'(\omega)\left[B(0) + \frac{3}{4}\frac{\ddot{B}(0)}{2!}\varepsilon_0^2 - \frac{65}{16}\frac{B^{(4)}(0)}{4!}\varepsilon_0^4 + \cdots \right]\right\} \\ & + \cos \omega t \left\{ G''(\omega) + A''(\omega)\left[B(0) - \frac{1}{4}\frac{\ddot{B}(0)}{2!}\varepsilon_0^2 + \frac{5}{8}\frac{B^{(4)}(0)}{4!}\varepsilon_0^4 - \cdots \right]\right\} \\ & + \sin 2\omega t \left\{ A''(\omega)\left[\frac{1}{2}\frac{\dot{B}(0)}{1!}\varepsilon_0 + \frac{1}{4}\frac{B^{(3)}(0)}{3!}\varepsilon_0^3 + \cdots \right]\right\} \\ & + \cos 2\omega t \left\{ A'(\omega)\left[-\frac{1}{2}\frac{\dot{B}(0)}{1!} - \frac{1}{2}\frac{B^{(3)}(0)}{3!} - \cdots \right]\right\} \\ & + A'(\omega)\left[\frac{1}{2}\frac{\dot{B}(0)}{1!}\varepsilon_0 + \frac{3}{2}\frac{B^{(3)}(0)}{3!}\varepsilon_0^3 + \cdots \right]\Bigg\} \\ & + \sin 3\omega t\{\ldots\} + \text{higher order terms,}\end{aligned} \quad \text{(B-1)}$$

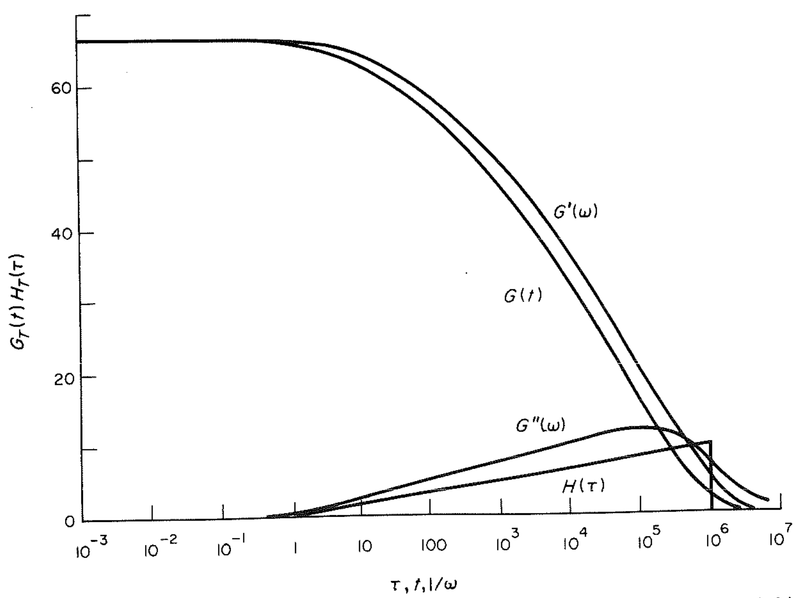

Fig. 5. Relaxation and dynamic moduli corresponding to the triangle spectrum $H(\tau)_{T[1,10^6]}$. Arbitrary units of stress on ordinate. Arbitrary units of time on abscissa.

where $\dot{B} = \frac{dB}{d\varepsilon}$, etc.; $B^n = \frac{d^nB}{d\varepsilon^n}$.

Now if the solid in question is subjected to a strain history containing many frequency components

$$\varepsilon(t) = \varepsilon_0 \sum_{n=1}^{N} a_n \sin \omega_n t,$$

the stress response may be written

$$\frac{\sigma(t)}{\varepsilon_0} = \sum_{n=1}^{N} \left\{ G_e \sin \omega_n t + \omega_n \int_{-\infty}^{t} [G(t-\tau, \varepsilon) - G_e] \cos \omega_n \tau \, d\tau a_n \right\}. \quad \text{(B-2)}$$

This is the sum of the responses to the individual frequency components; no interaction between these components occurs.

Consider now a solid describable by the Green–Rivlin series (equation 20). The frequency response of this type of solid has been treated by Lockett and Gurtin (1964); salient portions of their analysis are discussed below.

In linear viscoelasticity theory, the problem of dynamic response can be analyzed with the aid of the complex, one sided Fourier transform of the stress–relaxation function

$$G_1(t) = \hat{G}_1(\omega) = \int_0^{\infty} G_1(s) e^{-i\omega s} \, ds.$$

For the treatment of the nonlinear solid, a multiple transform may be defined analogously

$$\hat{G}(\omega_1, \omega_2, \ldots \omega_n) = \int_0^{\infty} \int_0^{\infty} \cdots \int_0^{\infty} G_n(s_1, s_2, \ldots s_n) \times e^{-i(\omega_1 s_1 + \omega_2 s_2 + \cdots + \omega_n s_n)} ds_1 \, d_2 \ldots ds_n. \quad \text{(B-3)}$$

This may be used to obtain from equation (20) a nonlinear constitutive equation in the frequency domain.

The problem of determining the solid's stress response to an oscillatory strain history

$$\varepsilon(t) = \sum_{n=1}^{N} a_n \sin \omega_n t$$

may be approached more directly by substituting the above history in equation (20) and making use of trigonometric identities (see also Lakes, 1975). The first and second terms of equation (20), called $\sigma_1(t)$ and $\sigma_2(t)$, become

$$\sigma_1(t) = \sum_{n=1}^{N} a_n [G'(\omega_n) \sin \omega_n t + G''(\omega_n) \cos \omega_n t], \quad \text{(B-4)}$$

$$\begin{aligned}\sigma_2(t) = \frac{1}{2} \sum_{n,p=1}^{N} a_n a_p \omega_n \omega_p \Big[& \operatorname{Re} \hat{G}_2(\omega_n, \omega_p) \cos(\omega_n + \omega_p)t \\ & - IM \hat{G}_2(\omega_n, \omega_p) t \sin(\omega_n + \omega_p)t \\ & + \operatorname{Re} \hat{G}_2(\omega_n, -\omega_p) \cos(\omega_n - \omega_p)t \\ & - IM \hat{G}_2(\omega_n, -\omega_p) \sin(\omega_n - \omega_p)t \Big]. \quad \text{(B-5)}\end{aligned}$$

The sums in the terms of order three and higher grow rapidly in complexity with the order of the term.