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Editor's Choice

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Shang Xinchun[1,2] and **Roderic S. Lakes**[2,3]

[1] Department of Mathematics and Mechanics, University of Science and Technology, Beijing, 30 College Road, Haidian, Beijing 10083, P.R. China
[2] Department of Engineering Physics, Engineering Mechanics and Materials Science Program, 1500 Engineering Drive, University of Wisconsin, Madison, WI 53706, USA
[3] Rheology Research Center, 1500 Engineering Drive, University of Wisconsin, Madison, WI 53706, USA

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Stability of cuboids and cylinders of an isotropic elastic material with negative stiffness under partial constraint is analyzed using an integral method and Rayleigh quotient. It is not necessary that the material exhibit a positive definite strain energy to be stable. The elastic object under partial constraint may have a negative bulk modulus K and yet be stable. A cylinder of arbitrary cross section with the lateral surface constrained and top and bottom planar surfaces is stable provided the shear modulus $G > 0$ and $-G/3 < K < 0$ or $K > 0$. This corresponds to an extended range of negative Poisson's ratio, $-\infty < \nu < -1$. A cuboid is stable provided each of its surfaces is an aggregate of regions obeying fully or partially constrained boundary conditions.

1 Introduction

Negative structural stiffness entails a reversal of the usual relationship between force and displacement in deformed objects: the applied force is in the opposite direction to the displacement rather than in the same direction. Negative structural stiffness can occur in objects with pre-strain including objects in the post-buckling regime [1]. Such objects contain stored energy and are unstable unless they are constrained. Negative stiffness is of interest in part because one can obtain extreme material damping in a system with elements of positive and negative stiffness. Such effects were observed experimentally [2] in a lumped system containing a post-buckled tube.

As for material stiffness, i.e. modulus, the condition of positive definite strain energy entails, for isotropic solids, $G > 0$ and $-1 < \nu < 0.5$ with ν as Poisson's ratio (see e.g. [3]). One can express this as $G > 0$ and $K > 0$ with K as bulk modulus. Positive definite strain energy implies stability for an elastic body under stress boundary conditions, including zero stress which means no constraint. A fully constrained object under displacement boundary conditions has a unique solution [4] and is incrementally stable [5] if the elastic moduli are strongly elliptic: for an isotropic solid, the criteria are $G > 0$ and $-\infty < \nu < 0.5$ or $1 < \nu < \infty$. Strong ellipticity requires the tensorial modulus C_{1111}, which governs the speed of plane compressional waves or the stiffness under compression with lateral constraint, to be positive. The bulk modulus can, however, be negative: $-4G/3 < K < \infty$.

[*] Corresponding author: e-mail: lakes@engr.wisc.edu, Fax: +1-608-263-7451

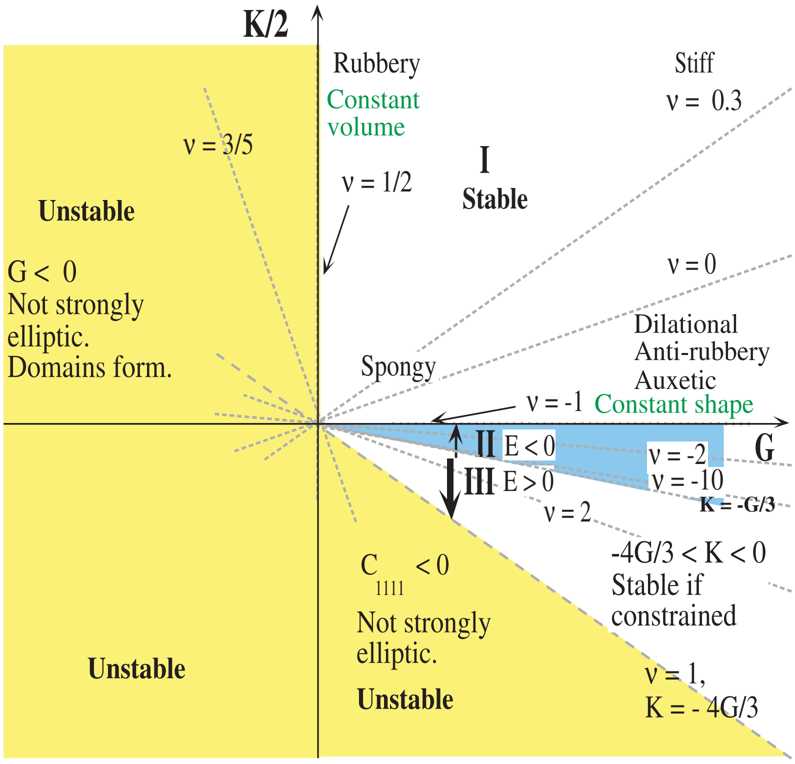

Fig. 1 (online colour at: www.pss-b.com) Map of bulk modulus K vs. shear modulus G, showing regions of negative Poisson's ratio ν and negative moduli for an isotropic solid.

Negative stiffness is distinct from negative Poisson's ratio. Poisson's ratio ν, is defined as the negative lateral strain of a stretched or compressed body divided by its longitudinal strain; it is dimensionless. Most materials stretched with longitudinal force elongate longitudinally but also contract *laterally*, hence have a positive Poisson's ratio. For most solids Poisson's ratio ranges between 0.25 and 0.33; the range for stability of isotropic solids is from -1 to 0.5; within that range all moduli are positive. Recently Lakes and co-workers have conceptualized, fabricated and studied negative Poisson's ratio foams [6, 7] with ν as small as -0.8. These materials become fatter in cross section when they are stretched and they are stable. Wojciechowski [8–10] analyzed several micro-structures predicted to exhibit negative Poisson's ratio. Milton [11] showed that negative Poisson's ratio can be achieved in hierarchical laminates, and that one can approach the isotropic lower limit -1 by proper choice of constituent moduli. Negative Poisson's ratio materials have been called anti-rubber by Glieck [12], dilational by Milton [11], and auxetic by Evans and co-workers [13, 14]. Negative stiffness, by contrast, refers to a situation in which a reaction force occurs in the same direction as imposed deformation. The relationship between the moduli and the Poisson's ratio, allowing negative values, is shown in Fig. 1. The upper right quadrant of this map was discussed by Milton to elucidate the role of negative Poisson's ratio in relation to the moduli.

Negative modulus is of interest in part because one can obtain extreme material damping [2, 15] in a composite with constituents of positive and negative modulus. In such a composite, the negative stiffness inclusions are under partial constraint from the surrounding matrix. The elastic and viscoelastic behavior of composites having a negative stiffness phase has been illustrated for composites with particulate ferroelastic inclusions [16]. Specifically, a composite with a tin matrix and a small concentration (1%) of vanadium dioxide particle inclusions was prepared. Vanadium dioxide was chosen since it exhibits a ferroelastic phase transformation at a convenient temperature 67 °C. This composite exhibited a large peak in mechanical damping and an anomaly in modulus in the vicinity of the transformation temperature. Composites with inclusions of negative stiffness may be called exterlibral or libral since they are on the boundary of balance [16]. Lakes and Drugan [17] studied spherically symmetric unit cells and showed selected solutions to be well behaved for inclusions of negative bulk modulus. Partial constraint is also of interest in the context of experimental study of materials in the vicinity of phase transformations. Negative bulk modulus [18] is possible in a pre-strained lattice and in several crystalline materials. Analysis of an Ising model [19] of a lattice predicts $K < 0$ near the critical temperature. Softening [20] of the bulk modulus (analogous to softening of the shear modulus in ferroelastics) has been observed in $YbInCu_4$ crystals at a temperature of 67 K. Composites with spherical inclusions of negative bulk moduli exhibit anomalies in the composite bulk modulus and Young's modulus (and in the corresponding mechanical damping). A partially constrained bar with negative bulk modulus is stable with respect to elementary deformation modes [18]. However, there are an infinite number of modes in a continuum. The stability of negative stiffness elastic media under partial constraint is not well understood. It is the purpose of the present work to explore the stability of isotropic elastic cuboids and cylinders under partial constraint of some surfaces.

2 Formulation

2.1 Governing equations for elastic solid

Consider an isotropic homogeneous elastic solid. The elastic body occupies a three dimensional region V in the Cartesian coordinate system (x, y, z), and it has a *regular* (bounding and piecewise smooth) surface ∂V. The motion of elastic body can be described by the displacements which dependent on the space coordinates (x, y, z) and the time t:

$$u_x = u_x(x, y, z, t)\,, \qquad u_y = u_y(x, y, z, t)\,, \qquad u_z = u_z(x, y, z, t)\,. \tag{2.1}$$

On the basis of classical theory of elasticity, the infinitesimal strains are given by

$$\begin{aligned} &\varepsilon_x = \frac{\partial u_x}{\partial x}\,, \qquad \varepsilon_y = \frac{\partial u_y}{\partial y}\,, \qquad \varepsilon_z = \frac{\partial u_z}{\partial z}\,, \qquad \gamma_{xy} = \left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right), \\ &\gamma_{yz} = \left(\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}\right), \qquad \gamma_{zx} = \left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right). \end{aligned} \tag{2.2}$$

The rotation is given by

$$\omega_x = \frac{1}{2}\left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right), \qquad \omega_y = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right), \qquad \omega_z = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right). \tag{2.3}$$

The stresses satisfy the linear isotropic elastic Hooke's law:

$$\begin{aligned} &\sigma_x = \lambda e + 2G\varepsilon_x\,, \qquad \sigma_y = \lambda e + 2G\varepsilon_y\,, \qquad \sigma_z = \lambda e + 2G\varepsilon_z\,, \\ &\tau_{xy} = G\gamma_{xy}, \qquad \tau_{yz} = G\gamma_{yz}\,, \qquad \tau_{xz} = G\gamma_{xz}\,, \end{aligned} \tag{2.4}$$

where the bulk strain $e = \varepsilon_x + \varepsilon_y + \varepsilon_z$. The Lame and shear moduli λ and G are related to Young's modulus E and Poisson's ratio ν by $\lambda = E\nu/[(1+\nu)(1-2\nu)]$, $G = E/[2(1+\nu)]$, respectively.

The displacement equation of motion ([21], p. 278) is

$$G\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(u, v, w) + (\lambda + G)\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = \rho\,\frac{\partial^2}{\partial t^2}(u, v, w)\,. \tag{2.5}$$

Suppose that the body forces are absent and the works of surface forces to displacements vanish at every point on the entire boundary of the elastic body, i.e. we have the boundary condition:

$$\sigma_{ij} n_j u_i = 0 \quad (i, j = x, y, z) \quad \text{on} \quad \partial V , \tag{2.6}$$

where the direction cosines $(n_x, n_y, n_z) = (\cos(\boldsymbol{n}, x), \cos(\boldsymbol{n}, y), \cos(\boldsymbol{n}, z))$ and $\boldsymbol{n}$ is the normal unit vector to the surface ∂V.

2.2 Dynamic stability and real frequency

In view of the intuitive notion of stability, an equilibrium configuration, or an equilibrium state, is stable, which means that the configuration will have only a small departure from its undisturbed configuration after any small disturbance. In the usual case, such a small disturbance may either induce small oscillation about the undisturbed equilibrium configuration or the perturbation becomes damped out. The well-known Liapounov's theory of dynamic stability provides fundamental methods and criteria to examine stability of an equilibrium state for *discrete* systems with a finite number of degrees of freedom. Liapounov's direct method in stability theory has been developed by Movchan to be more appropriate for application to *continuous* systems. Subsequently, Leipholz introduced Movchan's stability theorem for elastic body under conservative loads, and reached a conclusion that the stability of the equilibrium position is ensured by real and positive eigenvalue ω^2 of the problem [22]. In the case that the volume force and the surface load acting on the elastic body vanish, the eigenvalue ω^2 is namely the square of the natural frequency of the elastic body because small oscillatory motion superimposed on an initial natural state is free vibration. Leipholz's work is of considerable importance in theoretical analysis of stability as he established a connection between dynamic stability of natural state and natural frequency of elastic body. According to the aforementioned conclusion, if an elastic body has real natural frequency for all modes of free vibration then its natural state is stable. Therefore, we investigate stability conditions of the natural state for elastic materials, especially for elastic materials with negative stiffness. In this way we enable the determination of conditions under which the natural state is as a guide for experiments.

2.3 Rayleigh's Quotient

To investigate stability of an elastic body in the natural state, we consider first free vibration which is regarded as a small motion superimposed on an initial natural state of the elastic body. The displacements are assumed as

$$u_x = u(x, y, z)\, e^{i\omega t}, \qquad u_y = v(x, y, z)\, e^{i\omega t}, \qquad u_z = w(x, y, z)\, e^{i\omega t} , \tag{2.7}$$

where $i = \sqrt{-1}$ and ω is the resonant (natural) frequency. The displacement mode (2.7) is a *normal mode* in which all particles of the elastic body are moving synchronously, that is, passing through their rest positions simultaneously.

In the initial natural state, the displacements, stresses and the strains of an elastic body all vanish. The conservation of total energy requires that the sum of the kinetic energy and the potential energy become zero:

$$\int_V \left\{ \rho \left[\left(\frac{\partial u_x}{\partial t} \right)^2 + \left(\frac{\partial u_y}{\partial t} \right)^2 + \left(\frac{\partial u_z}{\partial t} \right)^2 \right] + W \right\} \mathrm{d}V = 0 , \tag{2.8}$$

where the volume mass density of material $\rho > 0$ and the strain energy function, i.e. the strain energy per unit volume, is given by ([21], p. 104)

$$W = \tfrac{1}{2}[(\lambda + 2G)(\varepsilon_x + \varepsilon_y + \varepsilon_z)^2 + G(\gamma_{xy}^2 + \gamma_{yz}^2 + \gamma_{zx}^2 - 4\varepsilon_x\varepsilon_y - 4\varepsilon_y\varepsilon_z - 4\varepsilon_z\varepsilon_x)] . \tag{2.9}$$

It can rewritten as

$$W = \tfrac{1}{6}\{(3\lambda + 2G)(\varepsilon_x + \varepsilon_y + \varepsilon_z)^2 + 3G(\gamma_{xy}^2 + \gamma_{yz}^2 + \gamma_{zx}^2) + 2G[(\varepsilon_x - \varepsilon_y)^2 + (\varepsilon_y - \varepsilon_z)^2 + (\varepsilon_z - \varepsilon_x)^2]\}. \quad (2.10)$$

Substituting (2.7) and (2.10) into the equation of energy conservation (2.8), we obtain the *Rayleigh's quotient*:

$$\omega^2 = R_0(u, v, w) = \frac{\int_V U_0 \, \mathrm{d}V}{\int_V \rho(u^2 + v^2 + w^2) \, \mathrm{d}V}, \quad (2.11)$$

where the function

$$U_0 = \frac{1}{6}\left\{(3\lambda + 2G)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)^2 + 3G\left[\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2\right] + 2G\left[\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y} - \frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial z} - \frac{\partial u}{\partial x}\right)^2\right]\right\}. \quad (2.12)$$

In variational methods of elastic vibration, Rayleigh's quotient plays an important role, for instance, Rayleigh's quotient provides an upper bound for the lowest eigenvalue: $\omega_{\min}^2 \le R$ [23]. In view of methods for energy, Rayleigh's quotient of an elastic body may contain all information of free vibration. It is more difficult to show stability than instability since there are an infinite number of modes. However, Rayleigh's quotient is an expression for the square of natural frequency ω^2 corresponding to *all* possible modes of free vibration; therefore it is given the same symbol. The meaning is to be understood by context. Moreover, the stability for elastic body is equivalent to that the Rayleigh's quotient having a positive lower bound [24]. Thus, Rayleigh's quotient could be a preferred approach to investigate stability of an elastic body. In general, the denominator of Rayleigh's quotient is always positive because it corresponds to the positive kinetic energy. In the case of conservative loads, the numerator of Rayleigh's quotient corresponds to the total potential energy, that is, the strain energy minus the work of external loads. Whether the numerator is positive or not is dependent from the material moduli, the shape of boundary and the boundary condition of the elastic body.

3 Stability under conditions of positive definiteness and strong ellipticity: review and analysis

3.1 Stability under the positive definiteness condition

From (2.10), the positive definiteness condition of the strain energy function W is

$$G > 0, \qquad 3\lambda + 2G > 0. \quad (3.1)$$

Obviously, this condition guarantees that the numerator of the Rayleigh's quotient (2.11) is positive since the function U_0 is positive definite. This implies that the square of natural frequencies for all modes of free vibration is positive, that is, the natural frequencies are real. It follows the well-known result: an elastic body with *arbitrary regular shape* made of positive definite material is always stable, whether the surface is constrained or not. Also the positive definiteness condition (3.1) requires all moduli to be positive. It excludes any possibility of negative modulus. In the following analysis, surface constraint is imposed and the stability of a solid of negative modulus under constraint is studied.

3.2 Stability of an elastic body fully constrained on the surface

Now we rewrite the strain energy function (2.9) as following Kelvin's form ([21], p. 168):

$$W = \frac{1}{2}\Bigg\{(\lambda+2G)\,(\varepsilon_x+\varepsilon_y+\varepsilon_z)^2 + 4G(\omega_x^2+\omega_y^2+\omega_z^2) + 4G\left[\left(\frac{\partial u_z}{\partial y}\frac{\partial u_y}{\partial z}-\frac{\partial u_y}{\partial y}\frac{\partial u_z}{\partial z}\right)+\left(\frac{\partial u_z}{\partial x}\frac{\partial u_x}{\partial z}-\frac{\partial u_x}{\partial x}\frac{\partial u_z}{\partial z}\right)+\left(\frac{\partial u_y}{\partial x}\frac{\partial u_x}{\partial y}-\frac{\partial u_x}{\partial x}\frac{\partial u_y}{\partial y}\right)\right]\Bigg\}. \tag{3.2}$$

Substituting (2.7) and (3.2) into the Eq. (2.8), we can obtain the another expression of the Rayleigh's quotient:

$$\omega^2 = R_1(u, v, w) = \frac{\int_V U_1\,\mathrm{d}V + I_1}{\int_V \rho(u^2+v^2+w^2)\,\mathrm{d}V}\,, \tag{3.3}$$

where the function

$$U_1 = \frac{1}{2}\left\{(\lambda+2G)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)^2 + G\left[\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)^2+\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)^2\right]\right\} \tag{3.4}$$

and the volume integral

$$I_1 = 4G\int_V \left[\left(\frac{\partial w}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial v}{\partial y}\frac{\partial w}{\partial z}\right)+\left(\frac{\partial u}{\partial z}\frac{\partial w}{\partial x}-\frac{\partial w}{\partial z}\frac{\partial u}{\partial x}\right)+\left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}\right)\right]\mathrm{d}V\,. \tag{3.5}$$

Using integration by parts, we can transform the volume integral (3.5) in V into the closed surface integral on ∂V as follows (to see A1. in the Appendix):

$$I_1 = 2G\oint_{\partial V}\Bigg\{n_x\left[\left(w\frac{\partial u}{\partial z}+\frac{\partial u}{\partial y}v\right)-u\left(\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\right]+n_y\left[\left(u\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}w\right)-v\left(\frac{\partial w}{\partial z}+\frac{\partial u}{\partial x}\right)\right] + n_z\left[\left(v\frac{\partial w}{\partial y}+\frac{\partial w}{\partial x}u\right)-w\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right]\Bigg\}\,\mathrm{d}S\,. \tag{3.6}$$

Obviously, for an elastic body with arbitrary regular shape if the material obeys the condition of *strong ellipticity*:

$$G > 0\,, \qquad \lambda + 2G > 0\,, \tag{3.7}$$

then we know from (3.4) that the function U_1 is non-negative. Indeed, a positive semi-definite form in the displacement gradients ([25], p. 122) is:

$$U_1 \ge 0\,. \tag{3.8}$$

Therefore, if the surface integral $I_1 = 0$, then the Rayleigh's quotient from (3.3) is *non-negative*. In Eq. (3.8), the equality holds only if the vector of displacement $\boldsymbol{u} = (u, v, w)$ has no dilatation and no rotation: $\nabla\cdot\boldsymbol{u} = 0$ and $\nabla\times\boldsymbol{u} = 0$, which implies that the vector of displacement is harmonic [26]:

$$\nabla^2\boldsymbol{u} = 0\,. \tag{3.9}$$

This equation has two situations of solution for the assumed homogenous boundary condition: either trivial solution $\boldsymbol{u} \equiv 0$ in V or non-trivial solution $\boldsymbol{u} \neq 0$ in V. The first situation guarantees that the Rayleigh's quotient is *positive definite*, so elastic body is *stable*. However, the latter leads to the possibility that Rayleigh's quotient (all frequency) becomes zero for some non-trivial solutions of displacement. In the latter situation, elastic body has *neutral stability*. Therefore, we obtain a result:

Conclusion I An elastic body with *arbitrary* regular shape which material moduli satisfy the condition of strong ellipticity (3.7) is either *stable* or *neutrally stable* if its boundary condition makes the surface integral (3.6) $I_1 = 0$. Moreover, under the assumed homogenous boundary condition, if Eq. (3.9) has a unique solution (the trivial solution $\boldsymbol{u} \equiv 0$), then the elastic body is *stable*; if the Eq. (3.9) has a non-trivial solution, then the elastic body is *neutrally stable*.

It should be pointed that the surface integral (3.6) $I_1 = 0$ in the Conclusion I is different from the Chen's integral boundary condition ([27], formula (13); [28], formula (A.1)). Chen's result indicates that for all fourth order skew tensor $\boldsymbol{W} = -\boldsymbol{W}^T$ if the displacement vector $\boldsymbol{u}$ satisfies the integral condition: $\boldsymbol{W} : \int_{\partial V} \boldsymbol{n} \otimes \nabla \otimes \boldsymbol{u} \otimes \boldsymbol{u} \, \mathrm{d}S = 0$ (where the symbol '$\otimes$' and '**:**' denote tensor product and double scalar product, respectively), then an elastic body with *arbitrary* regular shape which material moduli satisfy the condition of strong ellipticity is *stable*. As compared with Chen's integral condition, the integral condition $I_1 = 0$ in Conclusion I is more direct. However, in Conclusion I it requires the additional condition that the harmonic Eq. (3.9) has only a trivial solution $\boldsymbol{u} \equiv 0$ under the assumed homogeneous boundary condition. In some boundary conditions, this additional condition may hold. As a specific case of Conclusion I, when the boundary of the surface ∂V of an elastic body is fully constrained, *i.e.* the displacement boundary condition $u = v = w = 0$ on ∂V. In this case Eq. (3.9) has a unique trivial solution because of the uniqueness of the Dirichlet problem, and from (3.6) the surface integral $I_1 = 0$. Thus we reach another well-known result: an elastic body fully constrained (fixed) on the surface with arbitrary regular shape is stable if its material is strongly elliptic. It is worthwhile to note that the strong ellipticity condition (3.7) is weaker than the positive definiteness condition (3.1). The positive definiteness condition limits all moduli to be only positive, however the strong ellipticity condition allows some moduli to become negative.

3.3 Condition of strong ellipticity and negative elastic moduli

Strong ellipticity allows some moduli to be negative. Negative stiffness is not excluded by any physical law [17]. Examples of negative stiffness and the use of negative stiffness constituents are discussed in the Introduction.

In the view of physics, the strong ellipticity condition (3.7) is consistent with some restrictions upon materials. First, for isotropic elastic materials, the stress and strain share the same principal axes and the greater principal stress should occur in the direction of the greater principal strain. The two restrictions require that the empirical inequality and the Baker–Ericksen inequality in finite elasticity hold, respectively. In case of linear elasticity the two inequalities reduce to one of the conditions of strong ellipticity: $G > 0$ [29–31]. Also, another restriction on materials is that tension produces extension when the lateral faces are fixed; this requires that the tension-extension inequality in finite elasticity must be satisfied. In linear elasticity this inequality corresponds to another condition of strong ellipticity: $\lambda + 2G > 0$ [29, 31]. Moreover, the strong ellipticity condition (3.7) is a necessary and sufficient condition to ensure that the speeds of all plane waves in elastic media filling three-dimensional space are positive. Moreover, for the displacement boundary value problem of linear elastostatics in bounded regions, the strong ellipticity condition (3.7) guarantees the uniqueness of solution [26].

For isotropic elastic materials, the strong ellipticity condition (3.7) shows that the shear modulus G and tensorial compression modulus $C_{1111} = \lambda + 2G$ both are positive. If either of the moduli is negative, the material will undergo a deformation from its assumed initial state even though no forces are applied to it [32]. If strong ellipticity is violated, the material may exhibit an instability associated with the formation of bands of heterogeneous deformation [33]. Bands associated with negative C_{1111} were observed [34] in open cell foams under heavy compression. Bands which may be interpreted in terms of negative G are known in ferroelastic materials [35] below a critical phase transformation temperature. Based on these fundamental considerations, we investigate stability in the range $G > 0$ and $\lambda + 2G > 0$, i.e. strong ellipticity condition.

There exist three regions (Fig. 1) in the plane of elastic modulus G and K in which the strong ellipticity condition (3.7) holds:

(i) Region I: $G > 0\,, \qquad K > 0 \quad (E > 0, -1 < \nu < 1/2)\,.$

In this region, the material is positive definite and all elastic moduli are positive. For all boundary conditions, the material in this region is always stable. Region I corresponds to positive definite energy and is unquestionably stable. Negative Poisson's ratio, though counter-intuitive and not observed in common isotropic materials, is consistent with stability within the above range. Stability of elastic objects corresponding to the other regions is examined in the following.

(ii) Region II: $G > 0\,, \qquad -G/3 < K < 0 \quad (E < 0, -\infty < \nu < -1)\,.$

In this region both the Young's modulus E and the bulk modulus K are negative. Moreover, we have the condition:

$$G > 0\,, \qquad \lambda + G > 0\,. \tag{3.10}$$

(iii) Region III: $G > 0\,, \qquad -4G/3 < K < -G/3 \quad (E > 0, 1 < \nu < +\infty)\,.$

In this region negative bulk modulus K occurs, but Young's modulus E is positive.

Hence the condition of strong ellipticity allows the elastic material to have the negative bulk modulus K and the negative Young's modulus E. The stability of materials condition is not yet well understood in regions II and III.

3.4 Stability in partial constraint: Ryzhak's result

As discussed above, an elastic body with negative modulus for which all boundaries are fully constrained (fixed) is stable in the condition of strong ellipticity. By contrast, if its boundaries are fully stress-free, it is *usually* unstable even if the condition of strong ellipticity holds. The fully fixed and fully stress-free are the two extreme cases of boundary conditions. Stability conditions for other boundary condition cases between the two extreme cases are of interest and are studied in the present work.

By means of Fourier expansion procedure, Ryzhak [28] found a modified result: An elastic *cuboid* (rectangular parallelepiped) satisfying the condition of strong ellipticity (3.7) is stable if each of its surfaces has *one* of three types of boundary condition: (i) zero displacement (fixed); (ii) tangent directions to boundary are fixed but the normal direction is stress-free; (iii) normal is fixed but tangent stress-free.

The Ryzhak result implies that the full constraint of surface displacement is not necessary for a strongly elliptic material to be stable. In other words, it reveals a possibility of stability under partial constraint of the surface. In the following analysis, we will deal with the questions: Does there exist a *stable* or *neutrally stable* elastic body with negative bulk modulus (in the region II or III) if some portion of its boundary has not any constraint (stress-free)? A general answer of such question seems rather difficult. In the next two sections, we will give some positive answers to this question for elastic bodies with specific geometric shapes.

4 Stability of elastic cuboids with negative modulus

4.1 Simplification of the integral I_1

Consider an elastic cuboid with negative modulus and which obeys the strong ellipticity condition (3.7). The lengths of the cuboid are a, b, c in three dimensions, respectively. The six surfaces of the cuboid are denoted as:

$$\Omega_{1,2} = \{x = 0, a, 0 \le y \le b, 0 \le z \le c\}, \qquad \Omega_{3,4} = \{0 \le x \le a, y = 0, b, 0 \le z \le c\}, \\ \Omega_{5,6} = \{0 \le x \le a, 0 \le y \le b, z = 0, c\}. \tag{4.1}$$

On each surface of the cuboid, only the normal cosine is not zero and the two tangent cosines become zero:

$$n_x|_{\Omega_{1,2}} = \mp 1, \qquad n_y|_{\Omega_{3,4}} = \mp 1, \qquad n_z|_{\Omega_{5,6}} = \mp 1 \quad \text{(the others are zero)}. \tag{4.2}$$

In this case, the closed surface integral (3.6) can be rewritten by

$$I_1 = 2G\left\{\int_0^b\int_0^c \left[\left(\frac{\partial u}{\partial y} v + w\frac{\partial u}{\partial z}\right) - u\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right]\Bigg|_{x=0}^{x=a} \mathrm{d}y\,\mathrm{d}z \right. \\ + \int_0^c\int_0^a \left[\left(\frac{\partial v}{\partial z} w + u\frac{\partial v}{\partial x}\right) - v\left(\frac{\partial w}{\partial z} + \frac{\partial u}{\partial x}\right)\right]\Bigg|_{y=0}^{y=b} \mathrm{d}z\,\mathrm{d}x \\ \left. + \int_0^a\int_0^b \left[\left(u\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} v\right) - w\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right]\Bigg|_{z=0}^{z=a} \mathrm{d}x\,\mathrm{d}y\right\}.$$

Then, using integration by parts, we can reduce the integral I_1 into the following form:

$$I_1 = 2G\left\{-2\int_0^b\int_0^c u\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\Bigg|_{x=0}^{x=a} \mathrm{d}y\,\mathrm{d}z - 2\int_0^c\int_0^a v\left(\frac{\partial w}{\partial z} + \frac{\partial u}{\partial z}\right)\Bigg|_{y=0}^{y=b} \mathrm{d}z\,\mathrm{d}x - 2\int_0^c\int_0^a w\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\Bigg|_{z=0}^{z=c} \mathrm{d}x\,\mathrm{d}y + J\right\} \tag{4.3}$$

where the line integration is

$$J = \oint_{\partial R_{yz}} u(v\,\mathrm{d}z + w\,\mathrm{d}y)\Big|_{x=0}^{x=a} + \oint_{\partial R_{zx}} v(w\,\mathrm{d}x + u\,\mathrm{d}z)\Big|_{y=0}^{y=b} + \oint_{\partial R_{xy}} w(u\,\mathrm{d}y + v\,\mathrm{d}x)\Big|_{z=0}^{z=c}. \tag{4.4}$$

In the expression (4.4) the closed curves ∂R_{xy}, ∂R_{yz} and ∂R_{zx} are respectively the edges of the three rectangular regions which are denoted by

$$R_{xy} = \{0 \le x \le a, 0 \le y \le b\}, \qquad R_{yz} = \{0 \le y \le b, 0 \le z \le c\} \quad \text{and} \quad R_{zx} = \{0 \le z \le c, 0 \le x \le a\}$$

and located on the three different coordinate planes. Also all line integrals in (4.4) are along the counter-clockwise direction. Calculating these line integrals, we obtain the identity (see Appendix 2):

$$J \equiv 0. \tag{4.5}$$

4.2 Types of boundary conditions considered

On the premise of the boundary condition (2.6), the types of boundary sub-regions are classified into four groups. In a general case, each surface of the cuboid is an aggregation of the four types of sub-regions, namely

$$\Omega_i = B_u^{(i)} + B_{u\sigma}^{(i)} + B_{\sigma u}^{(i)} + B_\sigma^{(i)} \quad (i = 1, 2, \dots, 6)\,. \tag{4.6}$$

The four *independent* types of sub-boundary regions on the *i*-th surface Ω_i $(i = 1, 2, \dots, 6)$ are defined as follows:
Type I: The fixed boundary sub-region $B_u^{(i)}$. All components of displacement vanish in $B_u^{(i)}$:

$$u = v = w = 0 \quad \text{in} \quad B_u^{(i)}\,. \tag{4.7}$$

Type II: The mixed boundary sub-region $B_{u\sigma}^{(i)}$. The normal direction to boundary sub-region $B_{u\sigma}^{(i)}$ is fixed and the two tangent directions are stress-free. This type of boundary condition, for example, on the surface Ω_1 can be written as

$$u|_{x=0} = 0, \qquad \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\bigg|_{x=0} = \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\bigg|_{x=0} = 0 \quad \text{in} \quad B_{u\sigma}^{(1)}\,. \tag{4.8}$$

Type III: The mixed boundary sub-region $B_{\sigma u}$. The two tangent directions to boundary $B_{\sigma u}$ are fixed and the normal direction is stress-free. This type of boundary condition, for example, on the surface Ω_1 can be written by

$$\left[(\lambda + 2G)\,\frac{\partial u}{\partial x} + \lambda\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right]\bigg|_{x=0} = 0\,, \qquad v|_{x=0} = w|_{x=0} = 0 \quad \text{in} \quad B_{\sigma u}^{(1)}\,. \tag{4.9}$$

Type IV: The stress-free boundary sub-region B_σ. All stresses are free in B_σ. This type of boundary condition, for example, on the surface Ω_1 can be written as

$$\left[(\lambda + 2G)\,\frac{\partial u}{\partial x} + \lambda\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right]\bigg|_{x=0} = 0\,, \qquad \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\bigg|_{x=0} = 0\,, \qquad \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\bigg|_{x=0} = 0 \quad \text{in} \quad B_\sigma^{(i)}\,. \tag{4.10}$$

4.3 An extension of Ryzhak's result

Suppose that each surface of the cuboid is an aggregation of the type I, II and III boundary sub-regions, namely

$$\Omega_i = B_u^{(i)} + B_{u\sigma}^{(i)} + B_{\sigma u}^{(i)} \quad (i = 1, 2, \dots, 6)\,, \tag{4.11}$$

where boundary sub-regions $B_u^{(i)}$, $B_{u\sigma}^{(i)}$ and $B_{\sigma u}^{(i)}$ have at least one non-empty segment.

Taking into account that the partial derivative with respective to variable y or z can be exchanged in order with evaluation at $x = 0$, that is

$$\frac{\partial u}{\partial y}\bigg|_{x=0} = \frac{\partial}{\partial y}\,(u|_{x=0})\,, \qquad \frac{\partial u}{\partial z}\bigg|_{x=0} = \frac{\partial}{\partial z}\,(u|_{x=0})\,, \qquad \frac{\partial v}{\partial y}\bigg|_{x=0} = \frac{\partial}{\partial y}\,(v|_{x=0})\,, \qquad \frac{\partial w}{\partial z}\bigg|_{x=0} = \frac{\partial}{\partial z}\,(w|_{x=0})\,,$$

on the surface Ω_1 $(x = 0)$ we have boundary condition:

$$u|_{B_u^{(1)} \text{ or } B_{u\sigma}^{(1)}} = 0\,, \qquad \frac{\partial}{\partial y}\,(v|_{B_{\sigma u}^{(1)}}) + \frac{\partial}{\partial z}\,(w|_{B_{\sigma u}^{(1)}}) = 0\,.$$

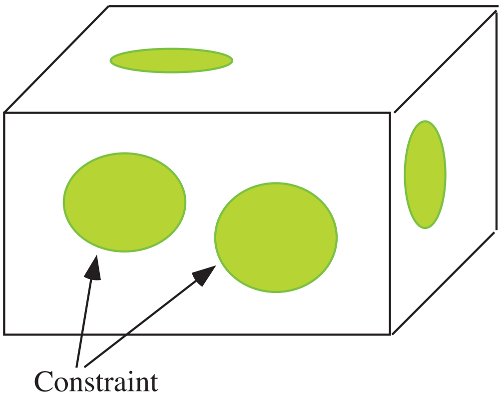

Fig. 2 (online colour at: www.pss-b.com) A cuboid constrained on its periphery so that each of its surfaces has at least one region of partial or full constraint.

Thus, the integral term on the surface Ω_1 $(x = 0)$ in the expression (4.3) becomes

$$\int_0^b \int_0^c u\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\Bigg|_{x=0} \mathrm{d}y\,\mathrm{d}z = \left(\iint\limits_{B_u + B_{u\sigma}^{(1)}} + \iint\limits_{B_{\sigma u}^{(1)}}\right)\left(u|_{x=0}\left[\frac{\partial}{\partial y}(v|_{x=0}) + \frac{\partial}{\partial z}(w|_{x=0})\right]\right)\mathrm{d}y\,\mathrm{d}z = 0\,. \tag{4.12}$$

Similarly, the all other integral terms in the expression (4.3) also vanish. In this case the integral $I_1 = 0$.

Hence the Rayleigh's quotient (3.3) is non-negative in the condition of strong ellipticity (3.7), since from the function U_1 is non-negative. Moreover, the positive definiteness of the function U_1 for displacements (u, v, w) is dependent on the uniqueness of solution of the Eq. (3.9). It is easy to see that except for the three cases of boundary condition, the Eq. (3.9) has unique zero solution if its each surface is an *aggregation* of the type I, II and III boundary sub-regions, i.e. (4.11) holds (proof to see A.3 in the Appendix). Thus, on the basis of Conclusion I, we obtain an extension of Ryzhak's result:

Conclusion II An elastic cuboid (Fig. 2) satisfying the condition of strong ellipticity (3.7) is *stable* if each of its surfaces is an *aggregation* of the type I, II and III boundary sub-regions, namely the condition (4.11) holds, except for three cases of boundary conditions which give *neutral* stability.

We remark that Ryzhak's result is for anisotropic elasticity; in that sense it is more general. However, it has a limitation that each surface of the cuboid is *only one* of the type I, II and III boundary sub-regions, but can not fit together in the three types of boundary sub-regions in the same surface, namely, it requires the limitation:

$$\Omega_i = B_u^{(i)} \quad \text{or} \quad B_{u\sigma}^{(i)} \quad \text{or} \quad B_{\sigma u}^{(i)} \quad (i = 1, 2, \dots, 6)\,. \tag{4.13}$$

Comparing with (4.11) it is clear that the condition (4.11) is weaker than the limitation (4.13). Distinctly, in the isotropic elasticity case, the aforementioned result extends Ryzhak's one. Actually, it has no limitation to the shape, number and location of the sub-regions $B_u^{(i)}$, $B_{u\sigma}^{(i)}$ and $B_{\sigma u}^{(i)}$ in each of the surfaces Ω_i $(i = 1, 2, \dots, 6)$.

4.4 Examples of exact solutions

In order to show that the square of natural frequency ω^2 is positive, we intend to give an exact solution to a specific example. Consider an elastic cuboid in which all six surfaces are type II ($\Omega_i = B_{u\sigma}^{(i)}$ $(i = 1, 2, \cdots, 6)$) and the condition of strong ellipticity holds. Although Ryzhak's *qualitative* result has covered the case of this example, however from application point of view, it is highly significant to find an exact solution of frequencies. To the end, for this example we assume displacement modes as

$$\begin{aligned} &u = A_1 \sin \alpha_l x \cos \beta_m y \cos \gamma_n z\,, \qquad v = A_2 \cos \alpha_l x \sin \beta_m y \cos \gamma_n z\,, \\ &w = A_3 \cos \alpha_l x \cos \beta_m y \sin \gamma_n z\,, \end{aligned} \tag{4.14}$$

where $\alpha_l = l\pi/a$, $\beta_m = m\pi/b$, $\gamma_n = n\pi/c$ $(l, m, n = 1, 2, \dots)$.

It is easy to validate that the displacement mode (4.14) satisfies all boundary conditions on the six surfaces. Substituting the displacement mode (4.14) into the equation (2.5) gives

$$[\boldsymbol{B} - \rho\omega^2 \boldsymbol{I}]\, \boldsymbol{A} = 0\ ,$$

where the vector $\boldsymbol{A} = [A_1, A_2, A_3]^T$ and the coefficient matrix

$$\boldsymbol{B} = \begin{bmatrix} (\lambda + 2G)\,\alpha_l^2 + G(\beta_m^2 + \gamma_n^2) & (\lambda + G)\,\alpha_l \beta_m & (\lambda + G)\,\alpha_l \gamma_n \\ & (\lambda + 2G)\,\beta_m^2 + G(\alpha_l^2 + \gamma_m^2) & (\lambda + G)\,\beta_m \gamma_n \\ \text{sym.} & & (\lambda + 2G)\,\gamma_n^2 + G(\alpha_l^2 + \beta_m^2) \end{bmatrix}.$$

The frequency equation is

$$f(\omega) = \det[\boldsymbol{B} - \rho\omega^2 \boldsymbol{I}\,] = -[\rho\omega^2 - G(\alpha_l^2 + \beta_m^2 + \gamma_n^2)]^2\, [\rho\omega^2 - (\lambda + 2G)\,(\alpha_l^2 + \beta_m^2 + \gamma_n^2)] = 0\,.$$

Hence we obtain

$$\omega_1^2 = \frac{\lambda + 2G}{\rho}\,(\alpha_l^2 + \beta_m^2 + \gamma_n^2) > 0\,, \qquad \omega_2^2 = \omega_3^2 = \frac{G}{\rho}\,(\alpha_l^2 + \beta_m^2 + \gamma_n^2) > 0 \quad (l, m, n = 1, 2, \ldots)\,. \tag{4.15}$$

As an another example, all surfaces are type III ($\Omega_i = B_{\sigma u}^{(i)}$ $(i = 1, 2, \ldots, 6)$), the displacement modes can be assumed as

$$\begin{aligned} &u = A_1 \cos \alpha_l x \sin \beta_m y \sin \gamma_n z\,, \qquad v = A_2 \sin \alpha_l x \cos \beta_m y \sin \gamma_n z\,, \\ &w = A_3 \sin \alpha_l x \sin \beta_m y \cos \gamma_n z\,. \end{aligned} \tag{4.16}$$

Similarly, the same solutions of frequencies are obtained as the expression (4.15).

4.5 Cuboid with stress-free surfaces: neutral stability

Now we consider an elastic cuboid in which the material satisfies the condition:

$$G > 0\,, \qquad \lambda + 2G = 0 \quad (\text{which is equivalent to } K = -\tfrac{4}{3}G < 0 \text{ or } E > 0 \text{ and } \nu = 1) \tag{4.17}$$

and the condition (4.6) holds, that is, each surface of the cuboid is an aggregation of the type I, II, II and IV boundary sub-regions, which includes stress-free boundary sub-region B_σ (Type 4). Observe that the condition on moduli and Poisson's ratio differs from the prior condition; a new result is under discussion. Under the condition (4.17), the boundary condition on sub-region B_σ, for example, on the surface Ω_1 $(x = 0)$ (4.10) can be reduced into

$$\left.\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right|_{x=0} = 0\,, \qquad \left.\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right|_{x=0} = 0\,, \qquad \left.\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right|_{x=0} = 0\,. \tag{4.18}$$

This leads to the integral term on the surface Ω_1 $(x = 0)$ in the expression (4.3) to vanish:

$$\begin{aligned} \int_0^b\!\!\int_0^c u \left.\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right|_{x=0} \mathrm{d}y\, \mathrm{d}z &= \left(\iint_{B_u^{(1)}} + \iint_{B_{u\sigma}^{(1)}} + \iint_{B_{\sigma u}^{(1)}}\right)\left(u|_{x=0}\left[\frac{\partial}{\partial y}(v|_{x=0}) + \frac{\partial}{\partial z}(w|_{x=0})\right]\right) \mathrm{d}y\, \mathrm{d}z \\ &\quad + \iint_{B_\sigma^{(1)}} \left.\left[u\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right]\right|_{x=0} \mathrm{d}y\, \mathrm{d}z = 0\,. \end{aligned}$$

Similarly, all other integral terms in the expression (4.4) also vanish, so the integral $I_1 = 0$.
It should be noticed that when the condition (4.17) holds the function U_1 is non-negative: $U_1 \geq 0$. The equality holds only if all rotations $\omega_x = \omega_y = \omega_z = 0$, that is $\nabla \times \boldsymbol{u} = 0$, it means that the displacements can be expressed by a potential function $\psi(x, y, z)$:

$$u = \frac{\partial \psi}{\partial x}, \qquad v = \frac{\partial \psi}{\partial y}, \qquad w = \frac{\partial \psi}{\partial z}. \tag{4.19}$$

Taking a special potential function

$$\psi = Cx^3 y^3 z^3 (x-a)^3 (y-b)^3 (z-c)^3 \tag{4.20}$$

it is evident that both displacements and the components of strain vanish in the six surfaces Ω_i $(i = 1, 2, \dots, 6)$, and it implies all components of stress vanish also in the six surfaces.

Thus, when each surface of the cuboid is *aggregation* of the type I, II, III and IV boundary sub-regions, the solution (4.19) and (4.20) is non-trivial solution of displacement in V. In this case the Rayleigh's quotient (3.3) degenerates into positive semi-definiteness but not positive definiteness. This leads to a result:

Conclusion III An elastic cuboid has *neutral stability* if the material satisfies the condition (4.17) and each of its surfaces is an *aggregation* of the type I, II, III and IV boundary sub-regions.

5 Stability of an elastic cylinder with negative modulus

Consider an elastic solid cylinder with arbitrary cross-section, not necessarily of circular or rectangular section, and arbitrary length; this includes cuboids under lateral constraint. Suppose that the lateral surface of the cylinder is fully fixed and the top and bottom surfaces are fully free, as shown in Fig. 3. Take the axis of shape-center of the cross section as the coordinate z-axis and the bottom surface in the xy-coordinate plane. Denote the lateral surface of the cylinder by D, the bottom and top surfaces by Ω_1 and Ω_2, respectively. The whole boundary of the cylinder is ∂V.

In order to examine the stability of the cylinder, we write the strain energy function (2.9) as new form:

$$W = \frac{1}{2}\left\{(\lambda + G)(\varepsilon_x + \varepsilon_y + \varepsilon_z)^2 + G\left[(\gamma_{zx}^2 + \gamma_{yz}^2 + 4\omega_z^2) + (\varepsilon_x + \varepsilon_y - \varepsilon_z)^2 + 4\left(\frac{\partial u_y}{\partial x}\frac{\partial u_x}{\partial y} - \frac{\partial u_x}{\partial x}\frac{\partial u_y}{\partial y}\right)\right]\right\}. \tag{5.1}$$

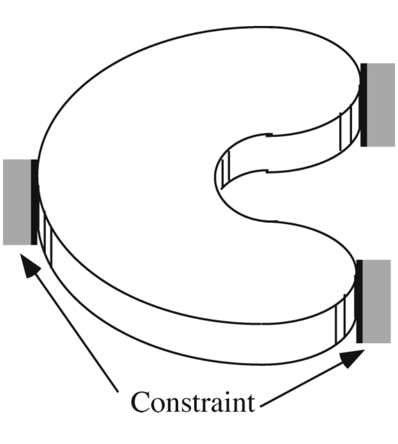

Fig. 3 A cylinder of arbitrary cross section constrained on its periphery (the lateral surface).

Substituting (5.1) into the strain-displacement Eq. (2.2), incorporating energy conservation, Eq. (2.8) we obtain the new form of Rayleigh's quotient:

$$\omega^2 = R_2(u, v, w) = \frac{\int\limits_V U_2 \, \mathrm{d}V + I_2}{\int\limits_V \rho(u^2 + v^2 + w^2)\, \mathrm{d}V} , \tag{5.2}$$

where the function

$$U_2 = \frac{1}{2}\left\{(\lambda + G)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)^2 + G\left[\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)^2\right] + G\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} - \frac{\partial w}{\partial z}\right)^2\right\} \tag{5.3}$$

and the integral

$$I_2 = 4G \int\limits_V \left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial y} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial y}\right) \mathrm{d}V .$$

The direction cosines on the top and bottom surfaces become

$$n_x\big|_{\Omega_{1,2}} = n_y\big|_{\Omega_{1,2}} = 0 \tag{5.4}$$

and the displacement boundary condition on the lateral surface is

$$u\big|_D = v\big|_D = 0 , \qquad w\big|_D = 0 . \tag{5.5}$$

Using integration by parts, we can transform the integral into the closed surface integral:

$$I_2 = 2G \int\limits_{D+\Omega_1+\Omega_2} \left[n_x\left(\frac{\partial u}{\partial y}\, v - u\,\frac{\partial v}{\partial y}\right) + n_y\left(u\,\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\, v\right)\right] \mathrm{d}S = 0 . \tag{5.6}$$

Obviously, under the condition (3.1) we have the function $U_2 \ge 0$. The equality holds only if the displacement u, v and w satisfy the following all equations:

$$\frac{\partial w}{\partial z} = 0 , \qquad \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = 0 , \qquad \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = 0 , \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 , \qquad \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 . \tag{5.7}$$

Integrating the front three equations in (5.7) we get the solution

$$w = \varphi(x, y) , \qquad u = -z\,\frac{\partial \varphi}{\partial x} + \psi_1(x, y) , \qquad v = -z\,\frac{\partial \varphi}{\partial y} + \psi_2(x, y) , \tag{5.8}$$

where φ, ψ_1 and ψ_2 are functions to be determined. Introducing the solution (5.8) into the last two equations in (5.7) and using the arbitrariness of the variable z we have

$$\nabla_2^2 \varphi = 0 \quad \left(\nabla_2^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \tag{5.9}$$

and Cauchy- Riemann equation of ψ_1 and ψ_2:

$$\frac{\partial \psi_1}{\partial x} + \frac{\partial \psi_2}{\partial y} = 0, \qquad \frac{\partial \psi_1}{\partial y} - \frac{\partial \psi_2}{\partial x} = 0.$$

This results in

$$\nabla_2^2 \psi_1 = 0, \qquad \nabla_2^2 \psi_2 = 0. \tag{5.10}$$

Substituting the solution (5.8) into the boundary condition (5.5), we get

$$\left.\frac{\partial \varphi}{\partial n}\right|_{\Gamma} = 0, \tag{5.11}$$

$$\psi_1|_{\Gamma} = 0, \qquad \psi_2|_{\Gamma} = 0, \tag{5.12}$$

where Γ is the closed line which is the boundary of the projective region of cross-section in the *xy*-coordinate plane. The uniqueness of the Dirichlet problem (5.10) and (5.12), and the Neumann problem (5.9) and (5.11) give $\psi_1 = \psi_2 \equiv 0$ and $\varphi = \text{constant}$, respectively. Also, from the last boundary condition in (5.5) and the first expression in (5.8), we have $\varphi|_{\Gamma} = 0$. Thus, $\varphi \equiv 0$. Hence, the function $U_2 \geq 0$. The equality holds only if $u = v = w = 0$. This implies that the Rayleigh's quotient (5.2) is positive definite. Here we obtain a new result:

Conclusion IV An elastic cylinder with arbitrary cross-section and in which the lateral surface is fully fixed and the top and bottom surfaces are fully stress-free is *stable* if its material satisfies the condition (3.1), namely $G > 0, \lambda + G > 0$.

As shown in Section 3, the (3.10) condition is allows $G > 0, -G/3 < K < 0$, i.e. that the elastic parameters are located in the region II in which both the Young's modulus E and the bulk modulus K are negative (see Fig. 1). We can also have $G > 0$, $K > 0$.

6 Discussion

In the elastic cuboid, there is no assumption regarding the proportions; the cuboid could be a long bar or a thin plate. It is therefore not surprising that a constraint, over all directions or some directions must be applied to a region of each surface for stability of a solid obeying strong ellipticity. Strong ellipticity allows negative bulk modulus K and negative Young's modulus E, but the lower modes of a long bar are governed by E. A fully free long bar therefore must therefore have a positive E to be stable. Stability conditions for cuboids with fixed proportions are not yet known. The present approach to the cuboid stability is on the basis of the Rayleigh's quotient and is totally different from the Ryzhak's Fourier expansion procedure. The present approach is more direct and allows us to deal with various much more complex assembled forms of the three types of boundary sub-regions in the same surface.

A particular case of the cuboid corresponds to neutral stability. Specifically, as an interesting example of the cuboid analysis, the elastic cuboid in which six surfaces are all stress-free is neutrally stable if its shear modulus $G > 0$ and bulk modulus $K = -4G/3 < 0$. Usually, an elastic body with any negative modulus is considered to be unstable if its boundaries are fully stress-free, however in this example neutral stability rather than instability occurs. The condition (4.17) leads to vanishing of the velocity of longitudinal waves of a medium. This condition was of some historical interest in the context of early models of electromagnetism based on a concept of space as an elastic medium [36]. For the special case that the entire boundary is fixed, Ericksen and Toupin [37] pointed that an elastic body with arbitrary regular shape has neutral stability when the condition (4.17) holds, and emphasized the importance of distinguishing between ordinary and neutral stability. Usually, the stability ensures the uniqueness of solution, however the neutral stability does not imply uniqueness of solution. Indeed, for the case of stress boundary and mixed boundary, the solution is non-unique under the condition [25, 26].

As for cylinders constrained on the lateral surface, the condition (3.10) satisfies the strong ellipticity condition (3.7). In the view of physics, the condition $\lambda + G > 0$ means the equibiaxial plane strain modulus is positive ([32], p. 107). As specific examples, the thin or thick elastic plates with the fixed side and the cuboid in which the top and bottom surfaces are fully stress-free and the other four surfaces are fully fixed are special cases of the cylinder discussed here. It is should be pointed out that the case of cuboid with two stress-free and four fixed surfaces is not covered in Ryzhak' result. Referring to Fig. 1, it is interesting that isotropic materials with Poisson's ratio below the lower stability limit of -1 can be stabilized by various kinds of constraint, but if Poisson's ratio exceeds the upper stability limit of 1/2, strong ellipticity is violated, leading to an instability which occurs regardless of surface constraint.

Implications regarding experiment are as follows. Negative structural stiffness is well known in the context of post-buckled elements, and has been observed experimentally (e.g. [2]). Negative elastic or viscoelastic moduli have been inferred from the behavior of composites containing ferroelastic inclusions, but have not been directly observed. The present results indicate the possibility of observation of negative moduli under constraint associated with instrumental displacement control.

7 Conclusions

It is not necessary that a partially constrained elastic solid exhibit a positive definite strain energy to be stable. The elastic object under partial constraint may have a negative bulk modulus and yet be stable. A cylinder of arbitrary cross section with the lateral surface constrained and top and bottom free surfaces is stable if the shear modulus $G > 0$ and $-G/3 < K < 0$ or $K > 0$. A cuboid is stable provided each of its surfaces is an aggregate of regions obeying fully or partially constrained boundary conditions.

Appendix

A1 Derivation of (3.6)

Using integration by parts, the first term in volume integral (3.5) becomes

$$\int_V \left(\frac{\partial w}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial v}{\partial y}\frac{\partial w}{\partial z} \right) \mathrm{d}V = -\int_V \left(\frac{\partial^2 w}{\partial y\,\partial z} v - v\frac{\partial^2 w}{\partial y\,\partial z} \right) \mathrm{d}V + \oint_{\partial V} \left(n_z \frac{\partial w}{\partial y} - n_y \frac{\partial w}{\partial z} \right) v\,\mathrm{d}S$$
$$= \oint_{\partial V} \left(n_z \frac{\partial w}{\partial y} - n_y \frac{\partial w}{\partial z} \right) v\,\mathrm{d}S$$

or another form:

$$\int_V \left(\frac{\partial w}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial v}{\partial y}\frac{\partial w}{\partial z} \right) \mathrm{d}V = -\int_V \left(w\frac{\partial^2 v}{\partial y\,\partial z} - \frac{\partial^2 v}{\partial y\,\partial z} w \right) \mathrm{d}V + \oint_{\partial V} \left(n_y \frac{\partial v}{\partial z} - n_z \frac{\partial v}{\partial y} \right) w\,\mathrm{d}S$$
$$= \oint_{\partial V} \left(n_y \frac{\partial v}{\partial z} - n_z \frac{\partial v}{\partial y} \right) w\,\mathrm{d}S\,.$$

For the other terms in volume integral (3.5) have similar expressions. These expressions result in (3.6).

A2 Derivation of the identity (4.5)

The line integration (4.4) can be rewritten as

$$J = 2G\left\{\left[\oint_{\partial R_{xy}} (wu)\Big|_{z=0}^{z=c}\,\mathrm{d}y + \oint_{\partial R_{yz}} (wu)\Big|_{x=0}^{x=a}\,\mathrm{d}y\right] + \left[\oint_{\partial R_{yz}} (uv)\Big|_{x=0}^{x=a}\,\mathrm{d}z + \oint_{\partial R_{zx}} (uv)\Big|_{y=0}^{y=b}\,\mathrm{d}z\right] + \left[\oint_{\partial R_{zx}} (vw)\Big|_{y=0}^{y=b}\,\mathrm{d}x + \oint_{\partial R_{xy}} (vw)\Big|_{z=0}^{z=c}\,\mathrm{d}x\right]\right\}. \tag{A2.1}$$

Calculating the front two terms in the above line integrals, we have

$$\begin{aligned}
&\oint_{\partial R_{xy}} (wu)\Big|_{z=0}^{z=c}\,\mathrm{d}y + \oint_{\partial R_{yz}} (wu)\Big|_{x=0}^{x=a}\,\mathrm{d}y \\
&= \int_0^b (wu)\Big|_{\substack{x=a\\z=c}}\mathrm{d}y - \int_0^b (wu)\Big|_{\substack{x=a\\z=0}}\mathrm{d}y + \int_b^0 (wu)\Big|_{\substack{x=0\\z=c}}\mathrm{d}y - \int_b^0 (wu)\Big|_{\substack{x=0\\z=0}}\mathrm{d}y \\
&\quad + \int_0^b (wu)\Big|_{\substack{x=a\\z=0}}\mathrm{d}y - \int_0^b (wu)\Big|_{\substack{x=0\\z=0}}\mathrm{d}y + \int_b^0 (wu)\Big|_{\substack{x=a\\z=c}}\mathrm{d}y - \int_b^0 (wu)\Big|_{\substack{x=0\\z=c}}\mathrm{d}y \\
&= 0\,.
\end{aligned}$$

Similarly, the other terms in the line integrals in (A.1) all vanish. Hence the identity $J \equiv 0$ is obtained.

A3 The proof for uniqueness of solution of the Eq. (3.9) in the condition (4.11)

On the basis of Green's first identity:

$$\int_V \boldsymbol{u}\cdot\nabla^2\boldsymbol{u}\,\mathrm{d}V = -\int_V |\nabla\boldsymbol{u}|^2\,\mathrm{d}V + \oint_{\partial V} \boldsymbol{u}\cdot\frac{\partial\boldsymbol{u}}{\partial n}\,\mathrm{d}S$$

and Eq. (3.9), we get

$$\int_V |\nabla\boldsymbol{u}|^2\,\mathrm{d}V = \oint_{\partial V} \boldsymbol{u}\cdot\frac{\partial\boldsymbol{u}}{\partial n}\,\mathrm{d}S \quad \left(\partial V = \sum_{i=1}^{6}\Omega_i\right). \tag{A3.1}$$

In the surface Ω_1 the integral

$$I_{\Omega_1} = \int_{\Omega_1} \boldsymbol{u}\cdot\frac{\partial\boldsymbol{u}}{\partial n}\,\mathrm{d}S = \int_{\Omega_1}\left(u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} + w\frac{\partial w}{\partial x}\right)\Big|_{x=0}\mathrm{d}y\,\mathrm{d}z\,.$$

Since equations $\nabla\cdot\boldsymbol{u} = 0$ and $\nabla\times\boldsymbol{u} = 0$ still hold in the boundary ∂V, we have

$$\frac{\partial u}{\partial x} = -\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right), \qquad \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}\,, \qquad \frac{\partial w}{\partial x} = \frac{\partial u}{\partial z}\,.$$

Thus the integral

$$I_{\Omega_1} = \int_{\Omega_1} \left\{ -u|_{x=0} \left[\frac{\partial}{\partial y} (v|_{x=0}) + \frac{\partial}{\partial z} (w|_{x=0}) \right] + \left[(v|_{x=0}) \frac{\partial}{\partial y} (u|_{x=0}) + (w|_{x=0}) \frac{\partial}{\partial z} (u|_{x=0}) \right] \right\} \mathrm{d}y\, \mathrm{d}z\,. \tag{A3.2}$$

Since the boundary condition in the surface $\Omega_1 = B_u^{(1)} + B_{u\sigma}^{(1)} + B_{\sigma u}^{(1)}$ is either $u = 0$ for $B_u^{(1)} + B_{u\sigma}^{(1)}$ or $v = w = 0$ for $B_{\sigma u}^{(1)}$, from (A3.2) we have $I_{\Omega_1} = 0$. Similarly, in the other surfaces the integral

$$\int_{\Omega_i} \boldsymbol{u} \cdot \frac{\partial \boldsymbol{u}}{\partial n}\, \mathrm{d}S = 0 \quad (i = 2, 3, \ldots, 6)\,.$$

Hence, from (A3.1) the displacement gradients all are zero. This implies the displacements are constants, namely, the rigid shift displacements. Only for the following three cases of boundary condition: (i) The surfaces $\Omega_{1,2}$ are type II and the others are type III; (ii) The surfaces $\Omega_{3,4}$ are type III and the others are type II; (iii) The surfaces $\Omega_{5,6}$ are type III and the others are type II, the rigid shift displacements are possible along x, y, z direction, respectively. Except for the three cases the Eq. (3.9) has unique zero solution.

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References

[1] J. M. T. Thompson, Instability and Catastrophes in Science and Engineering (John Wiley & Sons, London, 1982).
[2] R. S. Lakes, Philos. Mag. Lett. **81**, 95–100 (2001).
[3] S. P. Timoshenko and J. N. Goodier, Theory of Elasticity, 3rd edition (McGraw-Hill, New York, 1970).
[4] E. and F. Cosserat, C. R. Acad. Sci. Paris **126**, 1089–1091 (1898).
[5] Lord Kelvin (W. Thomson), Philos. Mag. **26**, 414–425 (1888).
[6] R. S. Lakes, Science **235**, 1038–1040 (1987).
[7] R. S. Lakes, Adv. Mater. **5**, 293–296 (1993).
[8] K. W. Wojciechowski, Mol. Phys. **61**, 1247–1258 (1987).
[9] K. W. Wojciechowski, Phys. Lett. A **137**, 60–64 (1989).
[10] K. W. Wojciechowski and A. C. Branka, Phys. Rev. A **40**, 7222–7225 (1989).
[11] G. W. Milton, Modelling the properties of composites by laminates, in: Homogenization and effective moduli of materials and media, edited by J. L. Erickson, D. Kinderlehrer, R. Kohn, and J. L. Lions (Springer Verlag, Berlin, 1986), pp. 150–175; J. Mech. Phys. Solids **40**, 1105–1137 (1992).
[12] J. Glieck, Anti-rubber, The New York Times (Science Times), 14 April 1987, p. 21.
[13] B. D. Caddock and K. E. Evans, J. Phys. D, Appl. Phys. **22**, 1877–1882 (1989).
[14] K. L. Alderson and K. E. Evans, Polymer **33**, 4435–4438 (1992).
[15] R. S. Lakes, Phys. Rev. Lett. **86**, 2897–2900 (2001b).
[16] R. S. Lakes, T. Lee, A. Bersie, and Y. C. Wang, Nature **410**, 565–567 (2001).
[17] R. S. Lakes and W. J. Drugan, J. Mech. Phys. Solids **50**, 979–1009 (2002).
[18] Y. C. Wang and R. S. Lakes, J. Compos. Mater. **39**, 1645–1657 (2005).
[19] D. J. Bergman and B. I. Halperin, Phys. Rev. B **13**, 2145–2175 (1976).
[20] B. Kindler, D. Finsterbusch, R. Graf, F. Ritter, W. Assmus, and B. Lüthi, Phys. Rev. B **50**, 704–707 (1994).
[21] A. E. H. Love, A Treatise on the Mathematical Theory of Elasticity, 4th edition (Dover, New York, 1944).
[22] H. H. E. Leipholz, Dynamic stability of elastic systems, in: Stability, Study No. 6, Solid Mechanics Division, edited by H. H. E. Leipholz (University of Waterloo, Waterloo, Canada, 1972), p. 256.

[23] L. Meirovitch, Methods of Analytical Dynamics (McGraw-Hill, New York, 1970), p. 493.
[24] R. J. Knops and E. W. Wilkes, Theory of elastic stability, in: Handbuch der Physik, Vol. III, edited by S. Flügge (Springer-Verlag, Berlin, 1973), p. 208.
[25] R. Hill, J. Mech. Phys. Solids **9**, 114–130 (1961).
[26] R. J. Knops and L. E. Payne, Uniqueness Theorems in Linear Elasticity (Springer-Verlag, New York, 1971), pp. 27–28.
[27] Y.-C. Chen, Arch. Ration. Mech. Anal. **113**, 167–175 (1991).
[28] E. I. Ryzhak, J. Mech. Appl. Math. **47**, 663–762 (1994).
[29] C. Truesdell and W. Noll, The non-linear field theories of mechanics, in: Handbuch der Physik, Vol. III, edited by S. Flugge (Springer-Verlag, Berlin, 1965), p. 159.
[30] M. F. Beatty, Appl. Mech. Rev. **40**, 1699–1734 (1987).
[31] W. Jaunzemis, Continuum Mechanics (The Macmillan Company, New York, 1967), pp. 434–435.
[32] R. S. Rivlin, Some thoughts on material stability, in: Finite Elasticity, edited by D. E. Carlson and R. T. Shield (Martinus Nijhoff Publishers, London, 1982), pp. 105–199.
[33] J. K. Knowles and E. Sternberg, J. Elast. **8**, 329–379 (1978).
[34] P. Rosakis, A. Ruina, and R. S. Lakes, J. Mater. Sci. **28**, 4667–4672 (1993).
[35] E. Salje, Phase transitions in ferroelastic and co-elastic crystals (Cambridge University Press, 1990).
[36] E. Whittaker, A History of the Theory of Aether and Electricity (Dover Publications Inc., New York, 1951), p. 144.
[37] J. L. Ericksen and T. Toupin, Canad. J. Math. **8**, 432–436 (1956); in: Foundations of Elasticity Theory, edited by C. Truesdell (Science Publishers, New York, 1965), pp. 432–436.