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Bounds on the complex bulk and shear moduli of a two-dimensional two-phase viscoelastic composite

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Abstract

The effective complex moduli of an isotropic two-phase, two-dimensional viscoelastic composite material are analyzed in terms of the complex moduli of its phases. The frequency range is assumed to be well below frequencies associated with the inertial terms; the acoustic wavelength is much larger than the inhomogeneities. Bounds are developed for the complex bulk modulus $K_* = K'_* + iK''_*$ and complex shear modulus $\mu_* = \mu'_* + i\mu'_*$ of the composite with *arbitrary* phase volume fractions. Shear modulus bounds are obtained subject to one scalar restriction on the phase properties $[(1/K_1 - 1/K_2)/(1/\mu_1 - 1/\mu_2)]'' = 0$ which is valid, in particular, for the phases with real and equal Poisson's ratios. Each of the moduli is shown to be constrained to a lens-shaped region bounded by two circular arcs in the complex bulk or shear modulus planes. The bounds are investigated numerically to explore conditions which give rise to high loss combined with high stiffness. Composite microstructures corresponding to various points on the circular arcs are identified. Influence of anisotropy of the composite on the stiffness-loss map for the bulk and shear type loads are analyzed.

1. Introduction

For applications of viscoelastic composite materials, one may desire stiff structural materials with high dissipation for the purpose of dumping vibration or reducing noise. Most stiff materials, however, are low in dumping and most high-dumping materials are compliant. Maximum dumping is limited by the fact that polymers, though they may exhibit high dumping, are not very stiff. One may combine stiff elastic and compliant, high-dumping materials in a composite to achieve the desired combination of the properties. For example, a layer of polymer can be cemented to a metal plate to increase the dumping, as is done in many cases to reduce vibration. Composite materials are in principle capable of high stiffness and high loss, provided there is non-affine deformation, in which the strain field is highly inhomogeneous on the scale of micro-structural elements, see e.g. Chen and Lakes (1993).

Alternatively, a small dissipation at low frequency, corresponding (via Fourier transformation) to minimal creep, may be considered useful for dimensional stability and to prevent creep buckling. A low value of dumping or creep can be achieved in composites which undergo affine deformation such as composites which approximate the Voigt model.

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In attempts to achieve design goals in applications, bounds on viscoelastic behavior combined with knowledge of microstructures which attain those bounds, can be useful in designing composite structural elements. Such bounds allow to predict the range of the properties of new composite materials without making costly experiments.

In this paper we study effective properties of a two-dimensional isotropic composite which is prepared from two isotropic viscoelastic materials taken in *arbitrary* proportions. Low frequency harmonic oscillation in such a medium can be described by the elasticity equations but with complex moduli and fields. The previous paper by Gibiansky and Lakes (1993) presented the complex bulk modulus bounds for a three-dimensional viscoelastic composite with arbitrary phase volume fractions. Here we obtain similar bounds for the two-dimensional problem, and also find bounds on the complex shear modulus (subject to additional constraint on the phase moduli that we will discuss later). These results were announced without proof in the paper by Gibiansky et al., 1993.

The method that we use here is identical to one used in the preceding paper by Gibiansky and Lakes (1993). Therefore, we concentrate on the results and omit details of the derivation. We base our analyses on the works by Gibiansky and Milton (1993a;b) who found bounds on the complex bulk and shear moduli of a two-dimensional composite with *fixed* volume fractions of the phases. The bounds on the complex bulk and shear moduli of a three dimensional composite with fixed volume fractions of the phases were found by Gibiansky and Milton (1993a) and Milton and Berryman (1996), respectively.

The properties of a two-dimensional isotropic viscoelastic material can be described by either of two pairs of moduli: plane Young's modulus E and plane Poisson's ratio ν or plane bulk modulus K and shear modulus μ. These pairs of the moduli are connected by the relations

$$E=\frac{4K\mu}{K+\mu},\qquad \nu=\frac{K-\mu}{K+\mu},\tag{1.1}$$

or

$$K=\frac{E}{2(1-\nu)},\qquad \mu=\frac{E}{2(1+\nu)}.\tag{1.2}$$

Note that for an isotropic two-dimensional material the bulk modulus K (in the plane strain problem) is connected with the three-dimensional bulk modulus κ and shear modulus μ by the formula

$$K=\kappa+\tfrac{1}{3}\mu,\tag{1.3}$$

whereas the shear modulus retains the same value as for the three-dimensional problem.

For the problem under study the bulk and shear moduli are complex and depend on frequency of oscillation. We assume that the frequency is fixed and is sufficiently low to allow the complex moduli description of the viscoelastic behavior. For a typical composite material with constituent dimensions less than 1 mm, this requirement entails frequencies below the MHz regime. Then the pair (K, μ) or (E, ν) completely characterizes the properties of an isotropic material at a given frequency. We denote K_1, K_2 and μ_1, μ_2 the bulk and shear moduli of the first and the second phases, respectively, $f_1=f$ and $f_2=1-f$ the phase volume fractions, and K_*, μ_* the effective complex bulk and shear moduli of a composite.

Restrictions on the dynamic viscoelastic functions can be derived from various energy principles, see Christensen (1972). For example, the requirement of a non-negative rate of energy dissipation gives the conclusion $K''\geq 0$, $\mu''\geq 0$, and $E''\geq 0$. Consider, for example, uniaxial strain field in the viscoelastic material. Since the dissipated energy per cycle is $\pi E''\epsilon^2$ (with ϵ as amplitude of the uniaxial strain), a negative value of E'' would correspond to a gain of mechanical energy per cycle. In passive materials, there is no external or internal source of energy which could supply this gain. Under these assumptions, we have $E''\geq 0$. Similarly, one can see that $K''\geq 0$ and $\mu''\geq 0$. Demonstration of the inequalities $K'\geq 0$, $\mu'\geq 0$, and $E'\geq 0$ requires the assumption of both non-negative stored energy and non-negative rate of energy dissipation. Restrictions on the complex Poisson's ratios can be obtained from the aforementioned conditions for the bulk and shear moduli. As for Poisson's ratio in a two-dimensional isotropic *elastic* material (i.e. material with real positive bulk and shear moduli), positiveness of the bulk and shear moduli requires that it lies in the interval $\nu\in[-1, 1]$ (in three dimensions, $\nu\in[-1, 0.5]$).

We are interested in finding the range of variation of the effective moduli K_*, μ_* when the microstructure of the composite and the phase volume fractions change. We restrict our attention to separate bounds on the complex bulk and shear moduli similar to the Hashin–Shtrikman–Walpole bounds for the elastic moduli. Such bounds can be presented either in the complex planes (K'_*, K''_*) and (μ'_*, μ''_*) or in stiffness-loss maps $(|K_*|, \tan\delta_K)$ and $(|\mu_*|, \tan\delta_\mu)$. Here $\tan\delta_K = K''_*/K'_*$ is the loss tangent, or tangent of the phase angle δ_K between stress and strain in sinusoidal hydrostatic loading, and $\tan\delta_\mu = \mu''_*/\mu'_*$ is the tangent of the phase angle δ_μ between stress and strain in sinusoidal shear loading.

For the shear modulus bounds we restrict our attention, following Gibiansky and Milton (1993b), to the special case when the ratio

$$g = \frac{1/K_1 - 1/K_2}{1/\mu_1 - 1/\mu_2}, \tag{1.4}$$

is real, i.e.,

$$g'' = 0. \tag{1.5}$$

Here and throughout the paper a', a'', and $|a| = \sqrt{(a')^2 + (a'')^2}$ denote the real, imaginary parts, and absolute value of the complex variable $a = a' + \mathrm{i}a''$, $\mathrm{i} = \sqrt{-1}$.

Eq. (1.5) is a technical assumption that allowed Gibiansky and Milton (1993b) to evaluate explicitly the bounds on the effective shear modulus of a two-phase composite *with fixed phase volume fractions*. Their bounds are valid, and our method is applicable to a general situation of complex ratio g, but the explicit form of the bounds for this general case is still not found. We note that Eq. (1.5) is satisfied, for example, for the phases with equal and real Poisson's ratios, i.e. when $\nu_1 = \nu_2 = \nu$, $\nu'' = 0$. In this case

$$g = \frac{\mu_1}{\kappa_1} = \frac{\mu_2}{\kappa_2} = \frac{1-\nu}{1+\nu}$$

obviously is real. When the ratios κ_1/μ_2 and κ_2/μ_1 are equal and real then

$$g = -\frac{\mu_1}{\kappa_2} = -\frac{\mu_2}{\kappa_1}$$

is also real.

The structure of the paper is as follows: in Section 2 we explain the basic idea how to obtain bounds on the effective moduli of a composite with *arbitrary volume fractions* and recapitulate the bounds by Gibiansky and Milton (1993a) and Gibiansky and Milton (1993b) on the effective moduli of a viscoelastic composite with *fixed phase volume fractions*. In Section 3 we obtain the bounds for the effective moduli of a composite with arbitrary phase volume fractions. In Section 4 we describe the structures of the composites which possess extremal viscoelastic moduli in a sense that they satisfy the bounds exactly. In Section 5 we apply our bounds to various situations. In particular, we analyze the bounds for the composite with stiff elastic and soft dissipative phases, study influence of the phase Poisson's ratios on the bounds, and analyze factors that give rise to the dissipation in the composite. In Section 6 we summarize results obtained.

A more detailed physical statement of the problem, description of the method and the structures, and the references on the earlier works on this subject can be found in Gibiansky and Milton (1993a) and Gibiansky and Lakes (1993).

2. Main idea of the method and known results

In this section we present a simple idea (Gibiansky and Lakes, 1993) how to bound effective moduli of a viscoelastic composite with *arbitrary phase volume fractions* given the bounds on the effective moduli of a composite with *fixed phase volume fractions*. Then we present the later bounds (Gibiansky and Milton, 1993a; Gibiansky and Milton, 1993b) in the form that is convenient for our use.

2.1. Bounding effective viscoelastic moduli

Let us illustrate the method on an example of the shear modulus bounds since the bulk modulus was discussed in detail in our earlier paper (Gibiansky and Lakes, 1993).

For any fixed phase 1 volume fraction f the effective shear modulus μ_* can be presented in a form

$$\mu_* = f\mu_1 + (1-f)\mu_2 - \frac{f(1-f)(\mu_1-\mu_2)^2}{(1-f)\mu_1 + f\mu_2 + y_{\mu_*}},$$
$$y_{\mu_*} \in Y_\mu. \tag{2.1}$$

Here Y_μ is some set in the complex plane which contains so-called Y-transformations y_{μ_*} (see Cherkaev and Gibiansky, 1992 and Milton, 1991)

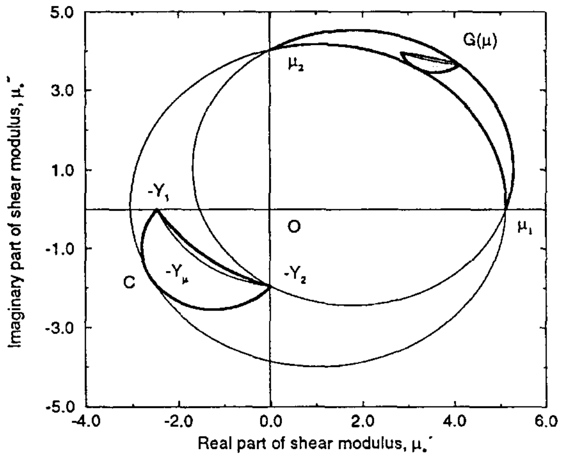
$$y_{\mu_*} = -f\mu_2 - (1-f)\mu_1 + \frac{f(1-f)(\mu_1-\mu_2)^2}{(1-f)\mu_2 + f\mu_1 - \mu_*} \tag{2.2}$$

of all composites with fixed phase volume fractions $f_1 = f$, $f_2 = 1-f$. There is a one-to-one correspondence (Eq. (2.1)) between μ_* and y_{μ_*}. Therefore, Y-transformation y_{μ_*} completely characterizes the shear modulus of a composite, given the phase moduli and volume fractions. For the problem under study the set Y_μ is formed by the outermost of several circular arcs (Gibiansky and Milton, 1993b). As we will see in the next section, the formulas for these arcs *do not depend on the phase volume fractions*. This is the key point of our analyses. Taking the Y_μ set as given, we seek to find the set $G(\mu)$ with includes the effective shear moduli of two-phase composites of all possible microstructures and phase volume fractions. In other words we need to find the union of all points μ_* that can be presented in the form of Eq. (2.1) where $f \in [0, 1]$ and $y_{\mu_*} \in Y_\mu$. In this section we describe how to solve this problem.

Let us denote $\mathrm{Arc}(\alpha_1, \alpha_2, \alpha_3)$ the arc of a circle in the complex α-plane that joins the points α_1, α_2 and when extended passes through the point α_3. One can check that such an arc is described by the point

$$\alpha = \gamma\alpha_1 + (1-\gamma)\alpha_2 - \frac{\gamma(1-\gamma)(\alpha_1-\alpha_2)^2}{(1-\gamma)\alpha_1 + \gamma\alpha_2 - \alpha_3}, \tag{2.3}$$

as parameter γ varies along the real axis in the interval [0, 1]. Comparing Eq. (2.3) with Eq. (2.1) one can see that for any fixed $y_{\mu_*} \in Y_\mu$ and $f \in [0, 1]$ the point μ_* defined by Eq. (2.1) draws an arc in the complex μ_*-plane that connects the points μ_1 and μ_2 of the original constituent materials and while extended passes through some point $-y_{\mu_*} \in -Y_\mu$. Here $-Y_\mu$ is the set of the points $-y_{\mu_*}$ such that $y_{\mu_*} \in Y_\mu$. In what follows we refer to Fig. 1a where the values of the moduli are chosen as $K_1 = 9.5$, $K_2 = 7.5i$ (in arbitrary units of stiffness), $\nu_1 = \nu_2 = 0.3$. Such values of the moduli look artificial but we chose them for illustration only. To find the



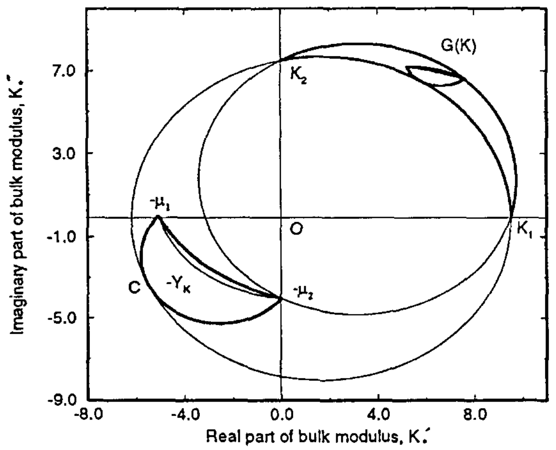


Fig. 1. The bounds for the shear (a) and the bulk (b) moduli of an isotropic composite of two isotropic phases of bulk moduli $K_1 = 9.5$, $K_2 = i7.5$, and Poisson's ratios $\nu_1 = \nu_2 = 0.3$. For a fixed volume fraction $f = 0.5$ the moduli K_* and μ_* are confined to the sets that are shaded. While f changes in the interval [0, 1] these sets draw the regions $G(K)$ and $G(\mu)$, respectively. The bounds of each of the sets $G(K)$ and $G(\mu)$ are given by the arcs of two circles. One of them contains the set $-Y_K$ or $-Y_\mu$, the other one has only one common point with it.

desired bounds on the set $G(\mu)$ of all possible μ_* defined by Eq. (2.1) we need to find the union of all the arcs that connect the points μ_1 and μ_2 and while extended cross the set $-Y_\mu$. Therefore the bounds for the $G(\mu)$ set are formed by the arcs of two circles. These arcs pass through the points μ_1 and μ_2 of the original constituent materials and when extended to circles touch the set $-Y_\mu$. One of these circles ($\mu_1 - C - \mu_2$ in Fig. 1a) contains this set, the other one ($\mu_1 - (-Y_2) - \mu_2$ in Fig. 1a) has only one common point $-Y_2$ with it.

The same arguments are valid for the bulk modulus bounds, see Fig. 1b. The only difference is in the definitions of the Y_μ set and corresponding Y_K set for the bulk modulus bounds. Moreover, bounds of this type are always formed by two arcs of the circles that pass through two points that corresponds to the phases, independently of the complexity of the Y-set shape. The only restriction is that the Y-set description should be independent of the phase volume fractions. Both the classical Hashin–Shtrikman variational method and the newer so-called translation method give the Y-set bounds that are independent of the phase volume fractions; see Gibiansky and Milton (1993a) for details and references. Variational principles that are a precondition for the use of variational methods were discovered for media with complex moduli by Cherkaev and Gibiansky (1994), see also Milton (1990).

2.2. *Complex bulk modulus of a composite with fixed phase volume fractions*

Bounds on the complex bulk modulus of a two-dimensional two-phase composite with fixed phase volume fractions can be described by the following statement:

Statement 1. (Gibiansky and Milton, 1993a): The Y-transformation y_{K_*} of the effective bulk modulus K_* of any two-phase viscoelastic composite belongs to the set Y_K bounded by the outermost pair of three circular arcs

$$y_K^{(1)}(\gamma) = \left(\frac{\gamma}{\mu_1} + \frac{1-\gamma}{\mu_2} \right)^{-1}, \tag{2.4}$$

$$y_K^{(2)}(\gamma) = -K_1 + \frac{(\mu_1 + K_1)(\mu_2 + K_1)}{\gamma(\mu_2 + K_1) + (1-\gamma)(\mu_1 + K_1)}, \tag{2.5}$$

$$y_K^{(3)}(\gamma) = -K_2 + \frac{(\mu_1 + K_2)(\mu_2 + K_2)}{\gamma(\mu_2 + K_2) + (1-\gamma)(\mu_1 + K_2)}, \tag{2.6}$$

where γ varies in the interval $\gamma \in [0, 1]$.

Eqs. (2.4), (2.5) and (2.6) define three arcs that pass through the points $y_K^{(n)}(0) = \mu_2$ and $y_K^{(n)}(1) = \mu_1$, $n = 1, 2, 3$, and while extended to circles pass the origin of the complex plane (the arc of Eq. (2.4)), the point $-K_1$ (the arc of Eq. (2.5)), and the point $-K_2$ (the arc of Eq. (2.6)), respectively. Comparing Eqs. (2.4), (2.5) and (2.6) with Eqs. (2.8), (2.9), (2.10) and (2.11) in the paper by Gibiansky and Lakes (1993) we see that the only difference with the three-dimensional bulk modulus bounds is that for the two-dimensional problem we need not take into account one of the arcs of Eq. (2.8) in Gibiansky and Lakes (1993) and the coefficient 4/3 is absent in front of μ_1 and μ_2 in Eqs. (2.4), (2.5) and (2.6). We will use this analogy to simplify the derivation.

In a recent paper Li and Weng (1994a) have observed that the bulk modulus bounds of statement 1 are nearly optimal: the complex bulk modulus of the two-phase composite with randomly oriented elliptical inclusions literally follows the bound when the aspect ratio of the inclusions changes from zero to one. Li and Weng (1994b) have found that the aforementioned three-dimensional bulk modulus bounds give sharp estimates on the complex bulk modulus of three-dimensional composites with spheroidal inclusions. They have used a Mori–Tanaka scheme to approximate the complex effective moduli.

2.3. *Complex shear modulus of a composite with fixed phase volume fractions*

Bounds on the complex shear modulus of a two-dimensional two-phase composite were found by Gibiansky and Milton (1993b) by using the translation method. Unfortunately, the above mentioned paper is still unpublished. The idea of the method and its application to the complex moduli bounds is explained by Gibiansky and Milton (1993a) using an example of the complex bulk modulus bounds. However, generalization for the shear modulus case is not trivial. We will not comment on the proof and will discuss only their results.

The bounds by Gibiansky and Milton (1993b) on the complex shear modulus of a composite with fixed phase volume fractions are presented as a system of inequalities on the effective complex shear modulus of a composite. The inequalities contain six free parameters that need to be optimized in order to get the best bounds. The nonlinear optimization problem for the bounds was solved in a particular case when the moduli of the initial material satisfy Eq. (1.5). We restrict our attention to this special but yet quite general case. The form of the shear modulus bounds differ depending on the sign of the real parameter g defined by Eq. (1.5). We will call the pair of the materials 'well-ordered' if this parameter is positive, i.e.

$$g' = \left[\frac{1/K_1 - 1/K_2}{1/\mu_1 - 1/\mu_2}\right]' \geq 0, \quad g'' = 0, \tag{2.7}$$

and we will call this pair 'badly-ordered' if the parameter g is negative, i.e.

$$g' = \left[\frac{1/K_1 - 1/K_2}{1/\mu_1 - 1/\mu_2}\right]' \leq 0, \quad g'' = 0. \tag{2.8}$$

A pair of materials is called well-ordered in the context of elasticity theory if both the bulk and shear moduli of one material are greater than bulk and shear moduli of the other one, respectively, i.e. if $(K_1 - K_2)(\mu_1 - \mu_2) \geq 0$. They are called badly-ordered in the opposite case when $(K_1 - K_2)(\mu_1 - \mu_2) \leq 0$ (see, e.g., Hashin, 1965; Walpole, 1966). Our definitions (Eqs. (2.7) and (2.8)) do not contradict the elastic theory definitions. Indeed, for the purely elastic phases (with real moduli) they coincide with the definitions of well- and badly-ordered material for the elastic theory. The only difference is that for an elastic material, the parameter g has a real value by definition whereas here we need to assume this condition in order to fix some ordering of the materials in a complex plane.

2.3.1. Composite with well-ordered phases

Statement 2 (Gibiansky and Milton, 1993b): The Y-transformation

$$y_{\mu_*} = -f\mu_2 - (1-f)\mu_1 + \frac{f(1-f)(\mu_1 - \mu_2)^2}{(1-f)\mu_2 + f\mu_1 - \mu_*} \tag{2.9}$$

of the effective shear modulus μ_*, of a two-dimensional two-phase viscoelastic composite with well-ordered phases subjected to the restriction (Eq. (1.5)), belongs to the set Y_μ^{w} which is bounded by the outermost pair of three circular arcs

$$y_\mu^{(1.\mathrm{w})}(\gamma) = \left(\frac{\gamma}{Y_1} + \frac{1-\gamma}{Y_2}\right)^{-1}, \tag{2.10}$$

$$y_\mu^{(2.\mathrm{w})}(\gamma) = -\mu_1 + \frac{(Y_1 + \mu_1)(Y_2 + \mu_1)}{\gamma(Y_2 + \mu_1) + (1-\gamma)(Y_1 + \mu_1)}, \tag{2.11}$$

$$y_\mu^{(3.\mathrm{w})}(\gamma) = -\mu_2 + \frac{(Y_1 + \mu_2)(Y_2 + \mu_2)}{\gamma(Y_2 + \mu_2) + (1-\gamma)(Y_1 + \mu_2)}. \tag{2.12}$$

Here γ varies in the interval $\gamma \in [0, 1]$ and

$$Y_1 = \frac{K_1 \mu_1}{K_1 + 2\mu_1}, \qquad Y_2 = \frac{K_2 \mu_2}{K_2 + 2\mu_2}. \tag{2.13}$$

Eqs. (2.10), (2.11) and (2.12) define three arcs that pass through the points $y_\mu^{(n.\mathrm{w})}(0) = Y_2$ and $y_\mu^{(n.\mathrm{w})}(1) = Y_1$, $n = 1, 2, 3$, and while extended to circles pass the origin of the complex plane (the arc of Eq. (2.10)), the point $-\mu_1$ (the arc of Eq. (2.11)), and the point (the arc of Eq. (2.12)), respectively. Comparing Eqs. (2.4), (2.5) and (2.6) and Eqs. (2.10), (2.11) and (2.12), one can see that they coincide up to the replacement of the symbols K_1, K_2, μ_1 and μ_2 in Eqs. (2.4), (2.5) and (2.6) by the symbols μ_1, μ_2, Y_1, and Y_2, respectively, in Eqs. (2.10), (2.11) and (2.12).

2.3.2. Composite with badly-ordered phases

Statement 3 (Gibiansky and Milton, 1993b): The Y-transformation y_{μ_*} of the effective shear modulus

μ_* of a two-dimensional two-phase viscoelastic composite with badly-ordered phases subjected to Eq. (1.5), belongs to the set Y_μ^b which is bounded by the outermost pair of three circular arcs

$$y_\mu^{(1.b)}(\gamma) = \left(\frac{\gamma}{Y_3} + \frac{1-\gamma}{Y_4}\right)^{-1}, \tag{2.14}$$

$$y_\mu^{(2.b)}(\gamma) = -\mu_1 + \frac{(Y_3+\mu_1)(Y_4+\mu_1)}{\gamma(Y_4+\mu_1)+(1-\gamma)(Y_3+\mu_1)}, \tag{2.15}$$

$$y_\mu^{(3.b)}(\gamma) = -\mu_2 + \frac{(Y_3+\mu_2)(Y_4+\mu_2)}{\gamma(Y_4+\mu_2)+(1-\gamma)(Y_3+\mu_2)}. \tag{2.16}$$

Here parameter γ varies in the interval $\gamma \in [0, 1]$ and

$$Y_3 = \frac{K_1\mu_2}{K_1+2\mu_2}, \qquad Y_4 = \frac{K_2\mu_1}{K_2+2\mu_1}. \tag{2.17}$$

Eqs. (2.14), (2.15) and (2.16) define three arcs that pass through the points $y_\mu^{(n.b)}(0) = Y_4$ and $y_\mu^{(n.b)}(1) = Y_3$, $n = 1, 2, 3$ and while extended to circles pass the origin of the complex plane (the arc of Eq. (2.14)), the point $-\mu_1$ (the arc of Eq. (2.15)), and the point (the arc of Eq. (2.16)), respectively. Again, replacement of the symbols K_1, K_2, μ_1, and μ_2 in (Eqs. (2.4), (2.5) and (2.6)) by the symbols μ_1, μ_2, Y_3, and Y_4, respectively, leads to Eqs. (2.14), (2.15) and (2.16).

3. Bounds on the complex moduli of a composite with arbitrary phase volume fractions

3.1. Bulk modulus bounds

Here we follow the idea presented in the Introduction. First, we put on the complex bulk modulus plane the points K_1, K_2, $-\mu_1$ and $-\mu_2$ and denote as O the origin of this plane, see Fig. 1b. Then we draw the arcs Arc($-\mu_1$, $-\mu_2$, O), Arc($-\mu_1$, $-\mu_2$, K_1) and Arc($-\mu_1$, $-\mu_2$, K_2). The outermost pair of these three arcs forms the $-Y_K$ set. The set $-Y_K$ is a set of the points $-y_{K_*}$ such that $y_{K_*} \in Y_K$ where Y_K is defined by statement 1. The set $-Y_K$ is bounded by two solid arcs in the lower left part of the Fig. 1b. These arcs and the $-Y_K$ set have no immediate physical meaning although they play an important role in the procedure that we describe. Now it is clear that the bounds on the complex bulk modulus of the composite with arbitrary volume fractions of the phases are given by two arcs (Arc(K_1, K_2, C) and Arc(K_1, K_2, $-\mu_2$) in Fig. 1b). They pass through the points K_1 and K_2 of the original constituent materials and when extended to circles touch the set $-Y_K$. The smaller shaded region inside the set $G(K)$ in Fig. 1b presents the bounds of statement 1 on the bulk moduli of the composite with fixed phase volume fractions $f_1 = f_2 = 0.5$. It consists of the points $K_*(y_{K_*})$ where the function $K_*(y_{K_*})$ is defined by Eq. (2.1), $y_{K_*} \epsilon Y_K$, and Y_K is described by statement 1.

Note two possible variants how the boundary circle may touch the $-Y_K$ set. It may pass through one of the corner points of this set, say $-\mu_2$ as in Fig. 1b or may touch the set $-Y_K$ in some internal point C of one of its boundary arcs.

Now we need only to solve a simple algebraic problem and define the formulas for these two circles as it was done in Gibiansky and Lakes (1993) for the three-dimensional problem. In fact we can use the solution given there, make obvious changes in notation (that were mentioned in Section 2.2), and get the following results:

Statement 4: To find the bounds on the effective complex bulk modulus of a two-dimensional two-phase viscoelastic composite one should calculate the values γ_*, and γ_{**},

$$\gamma_* = \begin{cases} (-B+\sqrt{D})/(2A), & \text{if } D \geq 0,\ A \neq 0 \\ -C/B, & \text{if } A = 0, \\ 2, & \text{if } D \leq 0, \end{cases} \tag{3.1}$$

$$\gamma_{**} = \begin{cases} (-B-\sqrt{D})/(2A), & \text{if } D \geq 0,\ A \neq 0 \\ -C/B, & \text{if } A = 0, \\ 2, & \text{if } D \leq 0, \end{cases} \tag{3.2}$$

where

$$A=\left[\frac{(1/\mu_1-1/\mu_2)}{(1/K_1-1/K_2)}\right]'',$$

$$B=\left[\frac{(1/K_1+1/K_2+2/\mu_2)}{(1/K_1-1/K_2)}\right]'', \tag{3.3}$$

$$C=\left[\frac{(1/K_1+1/\mu_2)(1/K_2+1/\mu_2)}{(1/K_1-1/K_2)(1/\mu_1-1/\mu_2)}\right]'',$$

$$D=B^2-4AC. \tag{3.4}$$

Then one should draw the arcs

$\mathrm{Arc}(K_1, K_2, -\mu_1)$, $\mathrm{Arc}(K_1, K_2, -\mu_2)$,

$$\mathrm{Arc}\left(K_1, K_2, -\left(\frac{\gamma_*}{\mu_1}+\frac{1-\gamma_*}{\mu_2}\right)^{-1}\right)$$

if $\gamma_* \in [0, 1]$,

$$\mathrm{Arc}\left(K_1, K_2, -\left(\frac{\gamma_{**}}{\mu_1}+\frac{1-\gamma_{**}}{\mu_2}\right)^{-1}\right)$$

if $\gamma_{**} \in [0, 1]$.

The outermost pair of these arcs gives the desired bounds for the $G(K)$ set, see Fig. 1b.

The expressions for the bounds can be simplified for the original materials that satisfy Eq. (1.5). It immediately follows from Eqs. (3.1), (3.2), (3.3) and (3.4) that in this case

$$\gamma_*=\gamma_{**}=-\frac{g(1+2R_K)}{2},$$

$$R_K=\left[\frac{1/K_2+1/\mu_2}{1/K_1-1/K_2}\right]'. \tag{3.5}$$

One can also check that for this case

$$\gamma_* \in [0, 1] \text{ if and only if } R_K \in \left[-\frac{1}{2}, -\frac{2+g}{2g}\right].$$

Therefore, we proved the following result:

Statement 5: To find the bounds on the effective bulk modulus of a composite of two phases with phase moduli that satisfy Eq. (1.5) one should calculate parameter γ_* by using Eq. (3.5). Then one should enscribe the arcs

$\mathrm{Arc}(K_1, K_2, -\mu_1)$,

$\mathrm{Arc}(K_1, K_2, -\mu_2)$,

$$\mathrm{Arc}\left(K_1, K_2, -\left(\frac{\gamma_*}{\mu_1}+\frac{1-\gamma_*}{\mu_2}\right)^{-1}\right)$$

if $\gamma_* \in [0, 1]$.

in the bulk modulus plane. The outermost pair of these arcs forms the desired bound for the $G(K)$ set, i.e. the bound for the complex bulk modulus of a viscoelastic composite with arbitrary phase volume fractions.

3.2. Shear modulus bound

Derivation of the shear modulus bounds is literally the same as we described in the previous subsection. Let us distinguish cases of well and badly ordered materials.

3.2.1. Composite with well-ordered phases

We define the set $-Y^{w}(\mu)$ as bounded by outermost pair of the circular arcs $\mathrm{Arc}(-Y_1, -Y_2, O)$, $\mathrm{Arc}(-Y_1, -Y_2, \mu_2)$ and $\mathrm{Arc}(-Y_1, -Y_2, \mu_2)$; see Fig. 1a. These arcs are described by the functions $-y_{\mu}^{(n,w)}$, $n=1, 2, 3$, respectively, while $\gamma \in [0, 1]$. The bounds on the complex shear modulus of the composite with arbitrary volume fractions of two well-ordered phases satisfying Eq. (1.5) are given by two arcs that pass through the points μ_1 and μ_2 of the original materials and when extended to circles touch the set $-Y^{w}(\mu)$. One of these circles contains this set, the other one has only one common point with it, see Fig. 1a.

Literally following the calculations for the bulk modulus bounds we obtain the following statement:

Statement 6: To find the bounds on effective shear modulus of a two-dimensional two-phase composite with well-ordered phases satisfying Eq. (1.5) one should calculate γ_*^{w},

$$\gamma_*^{w}=\frac{1+4R_\mu}{2(1+2g)}, \quad R_\mu=\left[\frac{1/K_2+1/\mu_2}{1/\mu_1-1/\mu_2}\right]', \tag{3.6}$$

and enscribe the arcs

$\mathrm{Arc}(\mu_1, \mu_2, -Y_1)$,

$\mathrm{Arc}(\mu_1, \mu_2, -Y_2)$,

$$\mathrm{Arc}\left(\mu_1, \mu_2, -\left(\frac{\gamma_*^{\mathrm{w}}}{Y_1} + \frac{1-\gamma_*^{\mathrm{w}}}{Y_2}\right)^{-1}\right)$$

if $\gamma_*^{\mathrm{w}} \in [0, 1]$

in the complex shear modulus plane. The outermost pair of these arcs forms the desired bounds.

Note that

$$\gamma_*^{\mathrm{w}} \in [0, 1] \text{ if and only if } R_\mu \in \left[-\frac{1}{4}, -\frac{3+4g}{4}\right].$$

3.2.2. Composite with badly-ordered phases

Literally following the calculations for the well-ordered materials we obtain the following statement.

Statement 7: To find the bounds on effective shear modulus of a two-dimensional two-phase composite with badly-ordered phases satisfying Eq. (1.5) one should calculate γ_*^{b}

$$\gamma_*^{\mathrm{b}} = \frac{3+4R_\mu}{2(1-2g)}, \tag{3.7}$$

(R_μ is defined by Eq. (3.6)) and enscribe the arcs

$\mathrm{Arc}(\mu_1, \mu_2, -Y_3)$,

$\mathrm{Arc}(\mu_1, \mu_2, -Y_4)$,

$$\mathrm{Arc}\left(\mu_1, \mu_2, -\left(\frac{\gamma_*^{\mathrm{b}}}{Y_3} + \frac{1-\gamma_*^{\mathrm{b}}}{Y_4}\right)^{-1}\right)$$

if $\gamma_*^{b} \in [0, 1]$

in the complex shear modulus plane. The outermost pair of these arcs forms the desired bounds.

Note that

$\gamma_*^{\mathrm{b}} \in [0, 1]$ if and only if

$$R_\mu \in \left[-\frac{3}{4}, -\frac{1+4g}{4}\right]. \tag{3.8}$$

4. Structures with extremal viscoelastic moduli

In this section we describe the structures that possess extremal viscoelastic properties, i.e lie on the boundary of sets defined by statements 3–7. Note that any of the arcs referred to in those statements may form the bound. Therefore, it is of interest to find the structures corresponding to the points on all of these curves to guide in the synthesis of materials with extremal properties. We will call a bound 'optimal' if there exists at least one composite that corresponds to any given point on this bound. Indeed, in this situation the bound cannot be improved without additional assumptions about the composite microstructure.

4.1. Structures with extremal bulk modulus

First we note that each point K_* on the $\mathrm{Arc}(K_1, K_2, -\mu_1)$ can be presented in the form

$$K_* = \gamma K_1 + (1-\gamma)K_2 - \frac{\gamma(1-\gamma)(K_1-K_2)^2}{(1-\gamma)K_1 + \gamma K_2 + \mu_1}, \tag{4.1}$$

where $\gamma \in [0, 1]$. Therefore, it corresponds to the effective bulk modulus of Hashin (1965) coated circles construction for some phase volume fractions $f_1 = \gamma$ and $f_2 = 1 - \gamma$. In these structures the inclusions of the second phase are surrounded by the first phase. Similarly each point on the $\mathrm{Arc}(K_1, K_2, -\mu_2)$ corresponds to the coated circles assemblages with the inclusions of the phase 1 in the matrix of

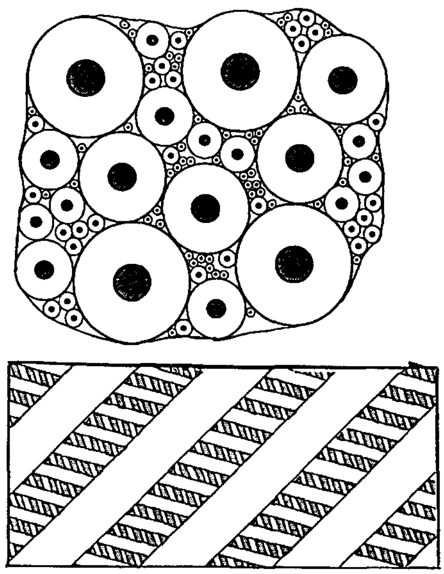


Fig. 2. Schematic picture of the optimal microstructures: (a) Hashin-type composite, (b) matrix laminate composite.

the phase 2. Therefore, two out of four arcs that are mentioned in statement 1 as candidates to form the bounds correspond to some known composites. If any of these arcs forms the bound this bound is optimal because there exist the composite materials that correspond to any point on that part of the boundary. The other microstructures that possess exactly the same bulk modulus are those of so-called matrix laminate composites. These microstructures were found to attain Hashin (1965) bulk and shear moduli bounds; see Francfort and Murat (1986). Therefore, the boundary $\mathrm{Arc}(K_1,\ K_2,\ -\mu_1)$ and $\mathrm{Arc}(K_1,\ K_2,\ -\mu_2)$ also correspond to known isotropic matrix laminate composites. Fig. 2 presents schematic pictures of both types of the extremal microstructures.

As to the other possible boundary arc

$$\mathrm{Arc}\left(K_1, K_2, -\left[\frac{\gamma}{\mu_1}+\frac{(1-\gamma)}{\mu_2}\right]\right) \text{ if } \gamma=\gamma_*$$

$$\text{or } \gamma=\gamma_{**} \text{ and } \gamma\in[0,1],$$

only one point on this arc is known that corresponds to some composite. Namely, there exists a composite that possesses bulk modulus K_* such that

$$y_{K_*}=\left[\frac{1-f}{\mu_1}+\frac{f}{\mu_2}\right]^{-1} \tag{4.2}$$

(see Gibiansky and Milton (1993a) for details). Therefore, if $\gamma\in[0,1]$ then the point

$$K_*=(1-\gamma)K_1+\gamma K_2 - \frac{\gamma(1-\gamma)(K_1+K_2)^2}{(1-\gamma)K_2+\gamma K_1+\left[\gamma/\mu_1+(1-\gamma)/\mu_2\right]^{-1}} \tag{4.3}$$

corresponds to the composite described by Eq. (4.2) when $f=1-\gamma$.

4.2. Structures with extremal shear modulus

One can check that any point μ_* on the $\mathrm{Arc}(\mu_1,\ \mu_2,\ -Y_1)$ can be presented in a form

$$\mu_*=\gamma\mu_1+(1-\gamma)\mu_2 - \frac{\gamma(1-\gamma)(\mu_1-\mu_2)^2}{(1-\gamma)\mu_1+\gamma\mu_2+K_1\mu_1/(K_1+2\mu_1)},$$

$$\gamma\in[0,1]. \tag{4.4}$$

This coincides with the expression for the Hashin (1965) bound on the effective shear modulus of an elastic material if $f_1=\gamma$ and $f_2=1-\gamma$. Francfort and Murat (1986) have found an isotropic matrix laminate composite that possesses such a shear modulus. This composite has the second phase as an inclusion phase and the first phase as a matrix one. Therefore, any point on the $\mathrm{Arc}(\mu_1,\ \mu_2,\ -Y_1)$ corresponds to a matrix laminate composite; see Fig. 2b for the schematic view of such a structure. Isotropic matrix laminate composites with the matrix of the second phase and inclusions of the first phase correspond to the points on the $\mathrm{Arc}(\mu_1,\ \mu_2,\ -Y_2)$. Therefore, two out of three arcs that are mentioned in statement 3 as candidates to form the bounds correspond to some composites. If any of these arcs forms the bound this bound is optimal and corresponds to the matrix laminate composites. At the moment we do not know any other structures that may lie on the boundary of the $G(\mu)$ sets.

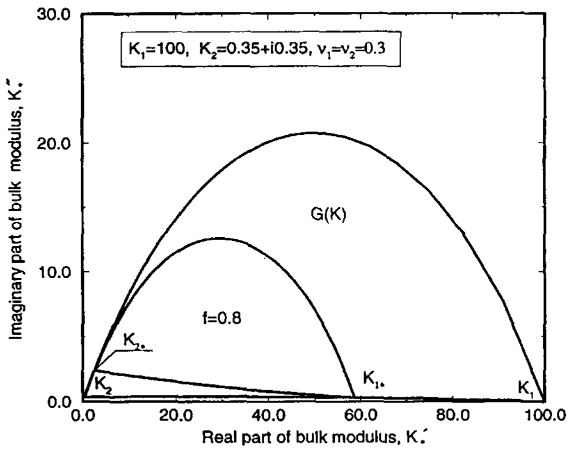


Fig. 3. The bounds for the bulk modulus of a composite of the stiff elastic phase and the soft viscoelastic phase. The lower bound corresponds to the assemblages of the Hashin coated circles. The upper bound nearly coincides with the curve that corresponds to the complementary Hashin coated circles. The internal smaller set shows the bulk modulus bounds for a composite with fixed volume fraction $f=0.8$ of the first phase.

5. Particular cases and discussion

In this section we apply our results to composites with specific phase moduli. Except for the example in Section 5.3 we restrict our attention to phases with real and equal Poisson's ratios. We evaluate the bounds and discuss results of numerical calculations.

5.1. Composite of stiff purely elastic phase and soft phase with high damping

Fig. 3 shows bulk modulus bounds of statement 5 for a composite material prepared from a stiff purely elastic phase with a bulk modulus $\kappa = 100$ and a soft phase with high damping with a bulk modulus $\kappa_2 = 0.35 + i0.35$ (in arbitrary units of stiffness). We assume that $\nu_1 = \nu_2 = 0.3$. In Fig. 3 we draw the $G(K)$ set that contains the effective complex bulk moduli K_* of all composites, and a smaller set that contains the bulk moduli K_* of composites with fixed volume fraction of the first phase $f = 0.8$. The lower bound is optimal and corresponds to the assemblages of Hashin (1965) coated circles where the stiff elastic phase 1 forms the matrix and the soft dissipative phase 2 is placed into the inclusions; the bulk modulus of such a composite with $f = 0.8$ is

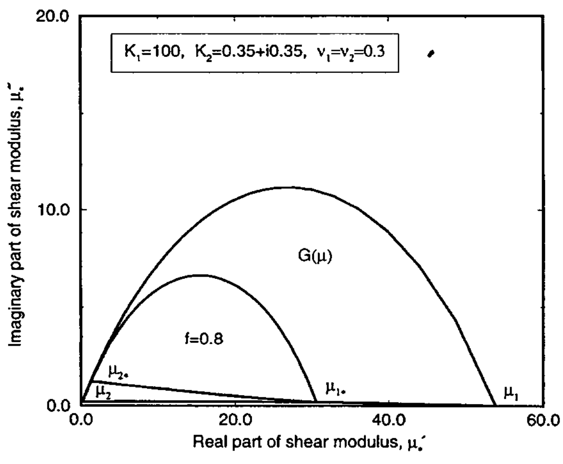


Fig. 4. The bounds for the shear modulus of a composites of a stiff elastic phase and a soft viscoelastic phase. The lower bound corresponds to assemblages of the matrix laminate composites. The upper bound nearly coincides with the curve that corresponds to the complementary matrix laminate composites. The internal smaller set shows the shear modulus bounds for a composite with fixed volume fraction $f = 0.8$ of the first phase.

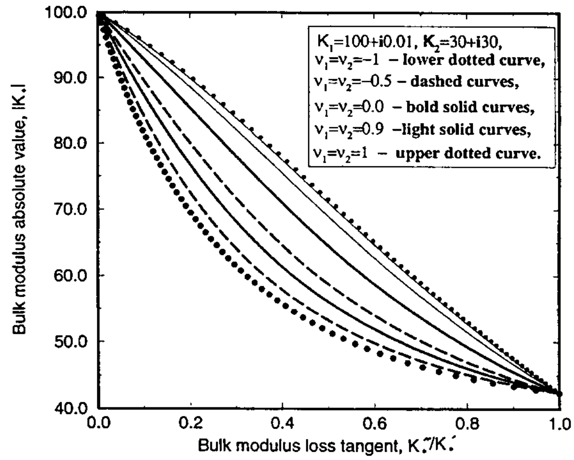


Fig. 5. The bulk modulus bounds for composites of phases with real and equal Poisson's ratios, $\nu_1 = \nu_2 = \nu$. The bounds are optimal and correspond to the assemblages of Hashin coated circles or to the matrix laminate composites.

denoted as K_{*1}. The upper bound nearly coincides with the curve that corresponds to the Hashin structures where the stiff inclusions of the phase 1 are surrounded by the soft phase 2; the bulk modulus of such a composite with $f = 0.8$ is denoted as K_{2*}. Although it cannot be seen in Fig. 3, the actual bound lies slightly above the curve that corresponds to the Hashin structures.

Fig. 4 shows the shear modulus bounds of statement 6 for a composite of the same phases. Here the lower bound is optimal and corresponds to the matrix laminate composites (see Francfort and Murat, 1986) where the stiff elastic phase 1 forms the matrix and the soft phase 2 is placed into the inclusions; the shear modulus of such a composite with $f = 0.8$ is denoted as μ_{1*}. The upper bound nearly coincides with the curve that corresponds to the matrix laminate composites where the stiff inclusions of the phase 1 are surrounded by the soft dissipative phase 2; the shear modulus of such a composite with $f = 0.8$ is denoted as μ_{2*}. The actual bound lies slightly above the curve that corresponds to these structures.

It is interesting to observe that a compliant phase with high damping can give rise to high damping in the composite in spite of the fact that the moduli of this phase are much smaller in absolute value than the moduli of the stiff elastic phase. This fact has been known for a long time; it may be explained by the high concentration of the strain field in the compliant phase under conditions in which much of the applied stress passes through the compliant phase. It is consistent with the results of Chen and Lakes (1993).

5.2. Composite of phases with equal and real Poisson's ratios

Figs. 5 and 6 show numerical results that describe dependence of the bulk and shear moduli bounds on the Poisson's ratio $\nu = \mu_1 = \nu_2$. To present the results in a form most helpful in evaluating practical usefulness of materials, we use stiffness loss maps in which the absolute value $|K''_*|K'_*$ (or $|\mu_*|$) of the bulk (or shear) modulus is plotted versus the loss tangent K'_*/K''_* (or μ''_*/μ'_*).

In the Fig. 5 example we take the phases with the bulk moduli $K_1 = 100 + i0.01$, $K_2 = 30 + i30$, and the Poisson's ratios $\nu = -1$ (lower dotted curve), $\nu = -0.5$ (dashed curves), $\nu = 0$ (bold solid curves), $\nu = 0.9$ (light solid curves) $\nu = 1$ (upper dotted curves). For these values of the parameters both the upper and the lower bounds are optimal and correspond to the Hashin assemblages of the coated circles.

It is interesting to observe that for the critical value of the Poisson's ratio $\nu = -1$, the shear moduli of the phases are equal to infinity, because in our example K is not equal to zero and

$$\mu = \frac{1-\nu}{1+\nu} K \tag{5.1}$$

for the plane elasticity problem. For the composite made of such materials the complex bulk modulus upper and lower bounds coincide, the effective properties are defined uniquely by the volume fractions of the phases and are independent of the microstructure. Namely, such a composite is isotropic with the shear modulus $\mu_* = \infty$ and the bulk modulus equals the arithmetic mean of the phase bulk moduli $K_* = fK_1 + (1-f)K_2$. This can be easily obtained by using the Hashin (1965) bounds on the shear modulus of an elastic composite, and elastic–viscoelastic correspondence principle (see Hashin, 1970 and Christensen, 1971).

It is interesting to observe that the arithmetic mean of the bulk moduli gives the *lower* bound in the stiffness-loss map, unlike the customary situation when the arithmetic mean provides an upper bound on the composite stiffness for given volume fraction. This concept was used by Brodt and Lakes (1995) in experiments to create composite laminates with high stiffness and high tan δ. It can be explained by the fact that a bound on stiffness corresponds to affine local deformation in the composite periodic cell, whereas the most efficient way to increase dissipation is to concentrate the strain field in the area occupied by the dissipative phase 2. As we will see, the upper bound on the dissipation is given by the harmonic mean averaging, that correspond to a homogeneous trial stress, or equivalently, to strain field which is concentrated in the soft dissipative phase 2.

Increasing of the phase Poisson's ratios ν while keeping bulk moduli fixed results in increasing the dissipation. If $\nu = 1$ then the shear moduli of both phases are equal to zero. Again, a composite of such phases is isotropic with the effective bulk modulus equal to the harmonic mean of the phase bulk moduli $K_* = (f/K_1 + (1-f)/K_2)^{-1}$, and the shear modulus $\mu_* = 0$. The upper and lower bulk moduli bounds coincide in this case, the effective properties are independent of the microstructure; they are shown by the upper dotted curve in Fig. 5. Note again the unusual property that the harmonic mean of the bulk

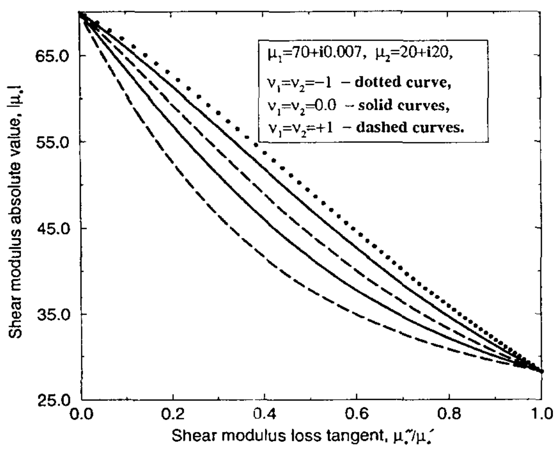


Fig. 6. The shear modulus bounds for composites of phases with equal and real Poisson's ratios, $\nu_1 = \nu_2 = \nu$. The bounds are optimal and correspond to the matrix laminate composites.

moduli gives the *upper* bound in the stiffness-loss map, that is in agreement with the observation by Brodt and Lakes (1995), and in contrast to the situation for the dependence of elastic properties upon volume fraction.

In the example in Fig. 6 we consider similar bounds for the shear modulus. We assume that $\mu_1 = 70 + i0.007$, $\mu_2 = 20 + i20$, the Poisson's ratios are equal $\nu_1 = \nu_2 = \nu$, where $\nu = -1$ (dotted curve), $\nu = 0.0$ (solid curves), and $\nu = 1$ (dashed curve). For these values of the parameters both the upper and the lower bounds are optimal and correspond to the matrix laminate composites.

For the critical value of the Poisson's ratio $\nu = -1$, the bulk moduli of the phases are equal to zero, because in this example μ is finite and

$$K = \frac{1+\nu}{1-\nu}\mu. \tag{5.2}$$

One can see that for the composite made of such materials the effective moduli of an isotropic composite are defined uniquely by the volume fractions of the phases and are independent of the microstructure. Namely, such a composite possesses the shear modulus equal to the harmonic mean of the phase shear moduli, $\mu_* = (f/\mu_1 + (1-f)/\mu_2)^{-1}$ the bulk modulus $K_* = 0$. Harmonic averaging again provides the upper bound on the effective shear modulus in the stiffness-loss map. Unlike the bulk modulus case, *increasing* of the phase Poisson's ratios while keeping shear moduli fixed results in *decreasing* the dissipation. In the opposite extreme case $\nu = 1$ the phase bulk moduli and the effective bulk modulus are equal to infinity. Nevertheless, the upper and lower bounds on the effective shear modulus do not coincide. They are given by the dashed curves in Fig. 6.

5.3. Composite of phases with complex Poisson's ratio

In this section we consider composites containing phases with complex values of the Poisson's ratios. Complex Poisson's ratios in constituent phases could be achieved by using appropriate materials, although the literature is sparse. Moreover, any experimental errors are magnified in calculating the Poisson's ratio from the complex bulk and shear moduli that

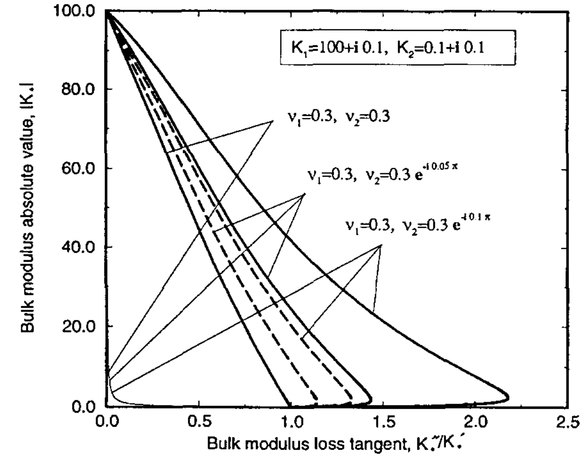


Fig. 7. The bulk modulus bounds for composites containing a stiff elastic phase and a soft dissipative phase with different complex Poisson's ratios. The light solid line is the optimal lower bound for all these cases, the bold solid curves are the upper bounds, the dashed curves correspond to the Hashin coated circles assemblages.

are more easily measured. The sign of the imaginary part ν'' can be positive or negative; correspondingly, in the time domain, the viscoelastic Poisson's ratio can increase or decrease, as shown by Lakes (1992).

We numerically evaluate the bulk modulus bounds for a composite containing phases with bulk moduli $\kappa_1 = 100 + i0.1$ and $\kappa_2 = 0.1 + i0.1$. Three cases are considered, all of which have $\nu_1 = 0.3$. The three values of Poisson's ratio of the second phase are $\nu_2 = 0.3$, $\nu_1 = 0.3\exp(-i0.05\pi)$, and $\nu_2 = 0.3\exp(-i0.1\pi)$. The results are presented in Fig. 7.

The light solid curve is the lower shear modulus bound for all these three cases. It corresponds to the Hashin–Shtrikman structures (assemblages of coated circles with the first phase forming the coating and the second phase forming the inclusions) and therefore is optimal. Note that in such structures the deformation field in the inclusions (i.e., in the phase 2 in this example) is hydrostatic if the applied field is hydrostatic. Therefore, effective bulk modulus is independent of the shear modulus (and therefore, on the Poisson's ratio) of the second phase, as we see in Fig. 7.

The bold solid curves show the upper bounds on the complex bulk modulus of these composites. For $\nu_2 = 0.3$ the curve corresponds to the Hashin–Shtrikman assemblages of coated circles with the second phase forming the coating and the first phase forming the inclusions. For the other two cases the upper bound has only one point that is known to be attainable; it is defined by Eq. (4.2). Two dashed curves correspond to the Hashin–Shtrikman structures with the inclusions of the first phase.

One can see that the loss tangent K''_*/K'_* of the composite can be greater than that of either phase. However there are two independent moduli (e.g. bulk and shear) and one can prove that the loss tangent associated with the bulk modulus of the composite can be no larger than the maximal and no smaller than minimal loss tangent (for the bulk or shear deformations) of either phase, i.e.

$$\min\left[\frac{K''_1}{K'_1}, \frac{K''_2}{K'_2}, \frac{\mu''_1}{\mu'_1}, \frac{\mu''_2}{\mu'_2}\right] \le \frac{K''_*}{K'_*} \le \max\left[\frac{K''_1}{K'_1}, \frac{K''_2}{K'_2}, \frac{\mu''_1}{\mu'_1}, \frac{\mu''_2}{\mu'_2}\right] \tag{5.3}$$

It is clear because the average dissipation rate of the composite cannot be smaller than the minimal or larger than the maximal dissipation rate of the phases. The upper bound for $\nu = 0.3\exp(-i0.1\pi)$, that allows the composites with a high dissipation rate is similar to the one described by Fig. 3. Indeed, such a choice of the Poisson's ratio leads to a small real part of the shear modulus of the second material. Together with the choice of the bulk moduli it guarantees the opportunity for high dissipation.

Observe that the upper bound varies little with the Poisson's ratio of the stiff phase. We do not show the figure that illustrates this because the upper bounds that correspond to different values of ν_1 cannot be distinguished in the scale of the figure.

We examine only bulk modulus bounds because the restriction $g'' = 0$ makes it difficult to change freely the Poisson's ratios and prevent us from isolating the effect of complex value of the Poisson's ratio in shear.

5.4. Factors that give rise to the loss tangent of the composite

In this section we analyze the conditions that give rise to the loss tangent of the composite. Fig. 8 shows our analyses of influence of the soft phase

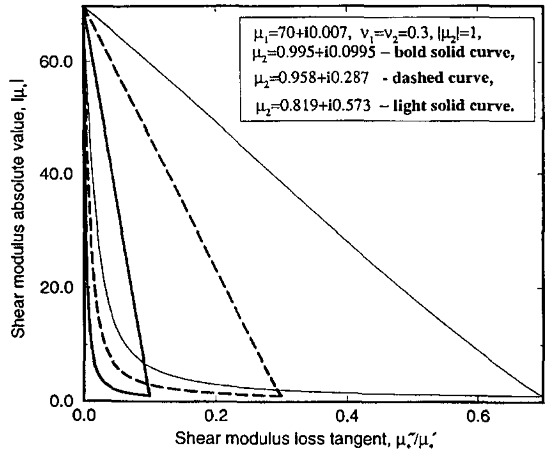


Fig. 8. The shear modulus bounds for composites containing a stiff elastic phase and a soft phase with different loss tangent but the same absolute value of the shear modulus. The bounds are optimal and correspond to the matrix laminate composites.

dissipation on the overall dissipation in the composite. We consider phases with the shear modulus of the first phase $\mu_1 = 70 + i0.007$, with $\tan\delta_\mu = 10^{-4}$, and Poisson's ratios $\nu_1 = \nu_2 = 0.3$. The second phase has shear modulus with absolute value 1 and different phase angles, i.e., $\mu_2 = 0.995 + i0.0995$, with $\tan\delta_\mu = 0.1$, (the bounds for this case are shown by the bold solid curves), $\mu_2 = 0.958 + i0.287$, with $\tan\delta_\mu = 0.3$, (the dashed curves), $\mu_2 = 0.819 + i0.573$, with $\tan\delta_\mu = 0.7$, (the light solid curves). One can see that an increase of the dissipation of the soft phase leads to a dramatic increase of the overall dissipation.

It is evident from Fig. 8 that increasing the stiffness of the stiff phase would lead to an increase of the overall dissipation, whereas small changes in the imaginary part of the stiff phase would not influence overall dissipation. These facts were also noted by Brodt and Lakes (1995).

5.5. Influence of anisotropy on the stiffness-loss map

In this section we examine how anisotropy influences the stiffness-loss map for the hydrostatic and shear deformations. To compare isotropic and anisotropic materials we note that the bulk and shear moduli of an *isotropic* composite can be expressed as follows

$$K_* = \tfrac{1}{2}\boldsymbol{a}_1:\boldsymbol{C}_*:\boldsymbol{a}_1,$$
$$\mu_* = \tfrac{1}{2}\boldsymbol{a}_2:\boldsymbol{C}_*:\boldsymbol{a}_2 = \tfrac{1}{2}\boldsymbol{a}_3:\boldsymbol{C}_*:\boldsymbol{a}_3. \tag{5.4}$$

Here

$$\boldsymbol{a}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \qquad \boldsymbol{a}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},$$
$$\boldsymbol{a}_3 = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \tag{5.5}$$

are the hydrostatic deformation and two orthogonal shear deformations, respectively, $\boldsymbol{C}_*$ is the fourth order complex stiffness tensor of an isotropic viscoelastic composite. The symbol (:) denotes contraction over two indices, i.e.

$$\boldsymbol{a}:\boldsymbol{C}:\boldsymbol{a} = \sum_{i,j,k,l=1}^{2} C_{jikl}a_{ij}a_{lk}. \tag{5.6}$$

We will compare our bounds for the complex bulk and shear moduli with similar expressions

$$K_L = \tfrac{1}{2}\boldsymbol{a}_1:\boldsymbol{C}_L:\boldsymbol{a}_1, \qquad \mu_L^{(1)} = \tfrac{1}{2}\boldsymbol{a}_2:\boldsymbol{C}_L:\boldsymbol{a}_2,$$
$$q\mu_L^{(2)} = \tfrac{1}{2}\boldsymbol{a}_3:\boldsymbol{C}_L:\boldsymbol{a}_3, \tag{5.7}$$

for the most anisotropic laminates composite with

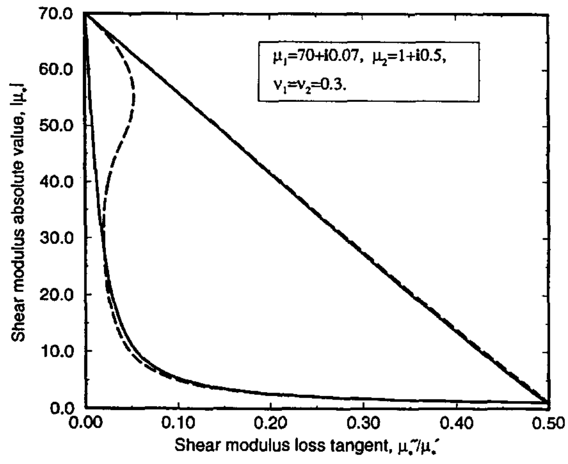


Fig. 10. The shear modulus bounds (the solid curves) and two shear modulus-type coefficients for the laminate composite (the dashed curves). The bounds are optimal and correspond to the matrix laminate composites.

stiffness tensor C_L. One can check (see e.g. Francfort and Murat, 1986) that

$$K_L = f_1K_1 + f_2K_2 - \frac{f_1f_2(K_1-K_2)^2}{f_2(K_1+\mu_1)+f_1(K_2+\mu_2)}, \tag{5.8}$$

$$\mu_L^{(1)} = f_1\mu_1 + f_2\mu_2 - \frac{f_1f_2(\mu_1-\mu_2)^2}{f_2(K_1+\mu_1)+f_1(K_2+\mu_2)}, \tag{5.9}$$

$$\mu_L^{(2)} = \left[\frac{f_1}{\mu_1}+\frac{f_2}{\mu_2}\right]^{-1}, \tag{5.10}$$

where the eigenvectors of the tensor $\boldsymbol{a}_2$ are directed along the laminates. Note that $\mu_L^{(2)}$ is equal to the harmonic average of the phase shear moduli. The complex constants K_L, and $\mu_L^{(1)}$ and $\mu_L^{(2)}$ describe the behavior of the laminate composite subjected to harmonic oscillation in the strain field proportional to the tensors $\boldsymbol{a}_1$, $\boldsymbol{a}_2$ and $\boldsymbol{a}_3$, respectively. These formulas give us an opportunity to compare anisotropic and isotropic materials.

Indeed, our complex bulk and shear moduli bounds restrict the values of the expressions (Eq. (5.4)) for isotropic composites. Eqs. (5.8), (5.9) and (5.10) present the complex values of similar expres-

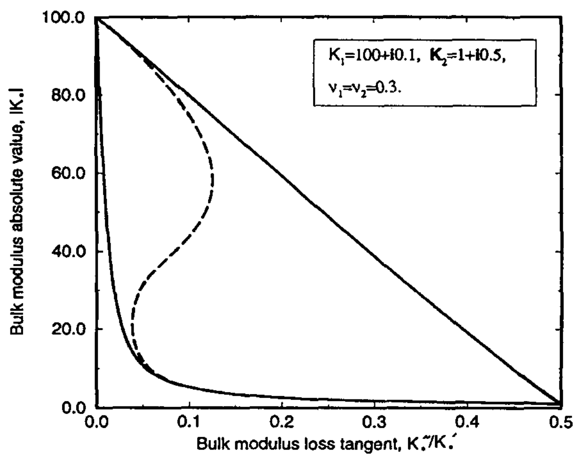


Fig. 9. The bulk modulus bounds (the solid curves) and the bulk modulus-type coefficient for the laminate composites (the dashed curve). The bounds are optimal and correspond to the Hashin structures.

sions (Eq. (5.7)) for laminate composites. Comparison of the data for the most anisotropic composite and the bounds that are valid for all isotropic composites illustrates the influence of anisotropy of the composite on the value of Eq. (5.4) of the coefficients of the complex effective stiffness tensor. We emphasize, however, that our bounds are applicable only to isotropic composites. As we will see, the values $\mu_L^{(1)}$ and $\mu^{(1)}$ will violate our bounds.

Figs. 9 and 10 present the comparison results. In Fig. 9 the solid curves show our volume-fraction independent bulk modulus bounds and the dashed curve shows the values K_L for the laminated composites containing phases with $K_1 = 100 + i0.1$, $K_2 = 1 + i0.5$, $\nu_1 = \nu_2 = 0.3$ and the volume fractions which vary in the admissible interval $f_1 = 1 - f_2 \in [0, 1]$. Note that the bounds are optimal in this case and corresponds to the Hashin (1965) coated circles assemblages with the inclusions of the stiff phase 1 (the upper bound) or the soft dissipative phase 2 (the lower bound). One can see that a laminate composite possesses intermediate dissipation under hydrostatic load. Indeed, such a load is isotropic and anisotropy of the composite cannot help to achieve better dissipation.

In Fig. 10 the solid curves show the isotropic shear modulus bounds and two dashed curves show the values $\mu_L^{(1)}$ (the lower dashed curve) and $\mu_L^{(2)}$ (the upper dashed curve) for the anisotropic laminate composite of the phases with $\mu_1 = 70 + i0.07$, $\mu_2 = 1 + i0.5$, $\nu_1 = \nu_2 = 0.3$. Note that the shear modulus bounds are also optimal for a composite of such phases and correspond to the matrix laminate composites with the inclusions of the stiff phase 1 (the lower bound) or the soft dissipative phase 2 (the upper bound). One can see that the laminate composite possesses extremal dissipation for the shear load of the $\boldsymbol{a}_3$ type, but the difference with the realizable upper bound is very small; the two curves nearly coincide in Fig. 10. Dissipation in the laminates is minimal for the shear load of the $\boldsymbol{a}_2$ type. Again, the laminate curve is pretty close to the lower bound for isotropic composites.

One can see that for isotropic or random load isotropic composites give better results, and even for the specific highly anisotropic load they nearly match the behavior of the most anisotropic laminates, at least for the phases that we examined.

6. Conclusions

In summary, in this paper we obtained the visual geometrical description and the explicit formulas for the bounds on the complex bulk and shear modulus of viscoelastic two dimensional two-phase composites with arbitrary volume fractions of the phases. The bounds have particularly simple form when the phases have equal and real Poisson's ratios. The bounds were analyzed in order to find the conditions that give rise to the dissipation rate of the composite. In particular, the known experimental fact that a small amount of soft dissipative inclusions in a stiff matrix may dramatically increase the overall dissipation rate of the composite, agrees with our bounds as can be seen in Figs. 3 and 4.

The results show that the lower and the upper bounds on the composite moduli differ significantly. Therefore, the microstructure of the composite material is of a great importance for the damping properties of the viscoelastic composites. The overall dissipation in the composite greatly depends on the dissipation in the soft phase, as illustrated in the figures; moreover the microstructure of the composite is of crucial importance, in harmony with Chen and Lakes (1993). Increasing the soft phase dissipative properties gives rise to a substantial increase in the overall dissipation. By contrast, a small dissipation in the stiff phase has little effect on the composite properties. Overall dissipation also increases with the stiffness of the stiff phase.

The dependence of the bounds on the Poisson's ratios of the phases was analyzed numerically; the results are summarized in Figs. 5 and 6 for the phase with real and equal Poisson's ratios, and in Fig. 7 for complex values of the soft phase Poisson's ratio.

The role of anisotropy in viscoelastic composites with arbitrary volume fraction differs from its role in elastic composites with fixed volume fraction. Introduction of anisotropy does not significantly expand the stiffness loss map, even for the favorable case of a single shear type load with prescribed axis. This is in contrast to the usual plots of stiffness versus volume fraction for elastic composites in which a substantial gain in stiffness for a given volume fraction can be achieved by introducing anisotropy.

The structures that correspond to the various parts of the bounds were identified. For most presented examples the bulk and shear moduli bounds are optimal and are attainable by the assemblages of Hashin coated circles (bulk modulus bounds) or by the matrix laminate composites (bulk and shear moduli bounds). The remaining bounds that are not shown to be optimal are rather restrictive although it is not known whether they can be improved or not.

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References

Brodt, M. and R.S. Lakes (1995), Composite materials which exhibit high stiffness and high viscoelastic damping, *J. Compos. Mater.* 29, 1823–1833.

Chen, C.P. and R.S. Lakes (1993), Analysis of high loss viscoelastic composites, *J. Mater. Sci.* 28, 4299–4304.

Cherkaev, A.V. and L.V. Gibiansky (1992), The exact coupled bounds for effective tensors of electrical and magnetic properties of two-component two-dimensional composites, *Proc. R. Soc. Edinburgh* 122 A, 93–125.

Cherkaev, A.V. and L.V. Gibiansky (1994), Variational principles for complex conductivity, viscoelasticity and similar problems for media with complex moduli, *J. Math. Phys.* 35(1), 1–19.

Christensen, R.M. (1971), *Theory of Viscoelasticity*, Academic Press, New York.

Christensen, R.M. (1972), Restrictions upon viscoelastic relaxation functions and complex moduli, *Trans. Soc. Rheol.* 16, 603–614.

Francfort, G. and F. Murat (1986), Homogenization and Optimal Bounds in Linear Elasticity, *Arch. Rat. Mech. Anal.* 94, 307–334.

Gibiansky, L.V. and R.S. Lakes (1993), Bounds on the complex bulk modulus of a two-phase viscoelastic composite with arbitrary volume fractions of the components, *J. Mech. Mater.* 16, 317–331.

Gibiansky, L.V., R.S. Lakes and G.W. Milton (1993), Viscoelastic composites with extremal properties, in: *Abstr. of the World Congr. Struct. Optimization 1998, Rio de Janeiro* 1, 218–225.

Gibiansky, L.V. and G.W. Milton (1993a), On the effective viscoelastic moduli of two phase media: I. Rigorous bounds on the complex bulk modulus, *Proc. R. Soc. Lond. A* 440, 163–188.

Gibiansky, L.V. and G.W. Milton (1993b), Shear modulus bounds for planar viscoelastic two-phase composite, unpublished manuscript.

Hashin, Z. (1965), On elastic behavior of fiber reinforced materials of arbitrary transverse phase geometry, *J. Mech. Phys. Solid* 13, 119–134.

Hashin, Z. (1970), Complex moduli of an viscoelastic composites: I. General theory and application to particulate composites, *Int. J. Solids Struct.* 6, 539–552; II. Fiber reinforced materials, *Int. J. Solids Struct.* 6, 797–807.

Lakes, R.S. (1992), The time dependent Poisson's ratio of viscoelastic cellular materials can increase or decrease, *Cell. Polym.* 10, 466–469.

Li, J. and G.J. Weng (1994a), Effective Creep Behavior and Complex Moduli of Fiber and Ribbon- Reinforced Polymer Matrix Composites, *Comp. Sc. Tech.* 52, 615–629.

Li, J. and G.J. Weng (1994b), Strain-rate sensitivity, relaxation behavior and complex moduli of a class of isotropic viscoelastic composites, *ASME J. Eng. Mat. Tech.* 116, 495–504.

Milton, G.W. and J,G. Berryman (1996), On the effective viscoelastic moduli of two phase media. II. Rigorous bounds on the complex shear modulus in three dimensions, *Proc. R. Soc. Lond. A.*, submitted.

Milton, G.W. (1990), On characterizing the set of possible effective tensors of composites: The variational method and the translation method, *Comm. Pure Appl. Math.* 43, 63–125.

Milton, G.W. (1991), The field equation recursion method, in: *Composite Media and Homogenization Theory*, eds. G. Dal Maso and G. Dell Antonio, Vol. 5, *Progress in Non-linear Differential Equations and their Applications*, series ed. H. Brezis, Birkhauser, Boston, pp. 223–245.

Walpole, L.J. (1966), On bounds for the overall elastic moduli of inhomogeneous systems-I, *J. Mech. Phys. Solids* 14, 151–162.