Mechanics Monograph (ASEE)
M-5 1-5 (1985)

Demonstration of Consequences of the Continuum Hypothesis

Lakes, R. S.,
"Demonstration of consequences of the continuum hypothesis",
Mechanics Monograph, M5 1-5 (1985).

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The significance of several crucial assumptions within the theory of deformable bodies can be demonstrated via simple observations of motion of a crack in a bar of a structured material. The demonstration is based on a null experiment of a sort requiring no instrumentation. The demonstration is therefore ideally suited for the classroom environment.

In developing concepts of stress and strain within an elementary course in the mechanics of deformable bodies, one relies heavily on arguments based on free-body diagrams of differential elements of material. Required assumptions, including those of continuity of the medium and the nature of the interactions across the boundary of the element, (i) are often not introduced at all, or (ii) are present implicitly only in the limiting process, or (iii) are soon forgotten. These assumptions are important not only in terms of a rational development of elementary mechanics, but also in view of modern uses of composite materials.

Consequences of the continuum hypothesis and its failure may be illustrated by means of a bar of flexible material, of square or rectangular cross section. Such a bar, if sufficiently large, is also useful for demonstrations of principal and anticlastic curvatures in bending, and warp of the cross sections in torsion. For the purpose of this article, consider two such bars, identical in size and shape. Let one be made of rubber and the other of polymeric foam such as that used as packing material.

Consider a differential element at the corner of the bar, as shown in figure 1. The surface traction upon the lateral surface is zero for torsional loading of the bar. By virtue of the symmetry of the stress tensor, the complementary shear stress must be zero, as shown in figure 1. Since all components of the shear stress vanish at the corner, the shear strain must also vanish. By such arguments one can show that the cross sections of a rectangular bar in torsion must undergo warp: see, e.g. F.P. Beer and E.R. Johnston, Jr., Mechanics of Materials, McGraw Hill, New York, 1981. Consider, however, a similar micro-element in a material with a lattice type structure, as shown in figure 2. The

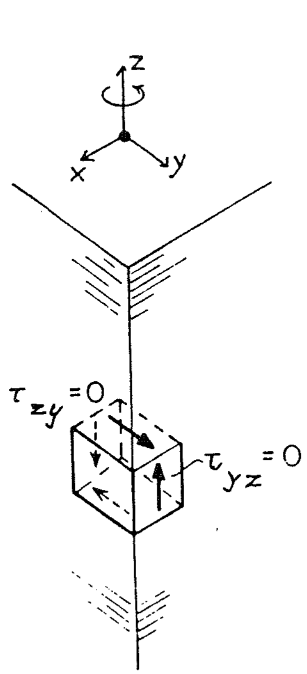

Fig. 1. Free-body diagram of corner element of unit width in an elastic continuum. Only those tractions involved in the rotational equilibrium of the element about the x axis are shown.

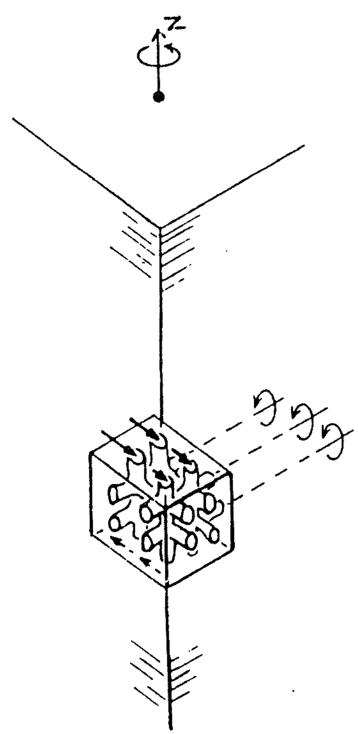

Fig. 2. Free-body diagram of corner element in a structure.

struts in the lattice can support a moment as well as a force, as shown. The moment must be nonzero since the transverse struts undergo a twist. For the micro-element to be in equilibrium, a nonzero shear force must be transmitted through the longitudinal struts as shown.

The corner elements of an elastic material and a lattice structure, then are predicted to behave quite differently. To illustrate this difference, make a small nick in the corner of the rubber bar and a similar nick in the corner of the foam bar. When the rubber bar is twisted, the crack is not expected to open in mode III (in which the faces of the crack shear parallel to each other and parallel to the crack front) since there is no stress in the region of the crack. While this is strictly true only for an infinitesimal crack, the crack opening is negligibly small if the crack is sufficiently short compared with the bar width, as shown in figure 3. The situation is different in the foam bar. The corner crack opens noticeably in mode III when the bar is twisted, as shown in figure 4. Here the corner of the bar has been made more visible by gluing black thread to it with

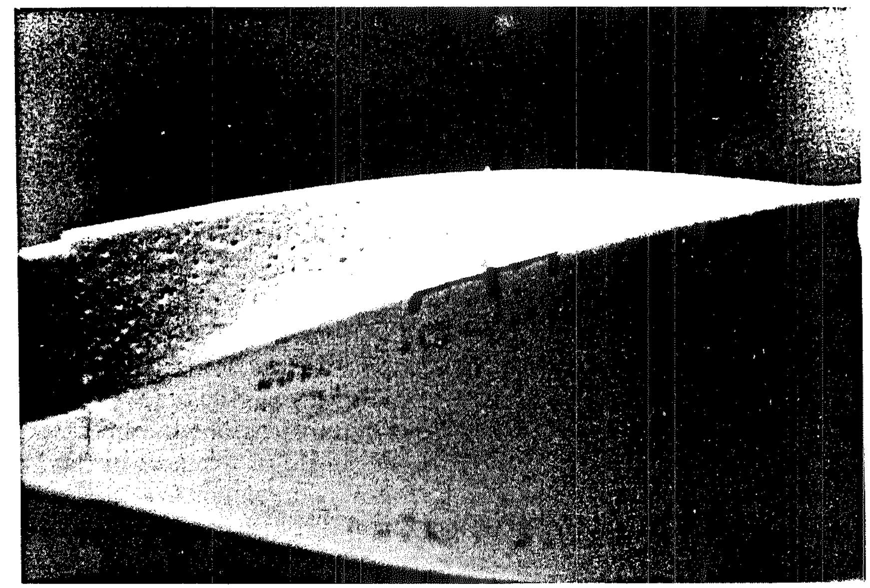

Fig. 3. Behavior of a crack in the corner of a bar of rubber in torsion.

Fig. 4. Behavior of a crack in the corner of a bar of foam in torsion.

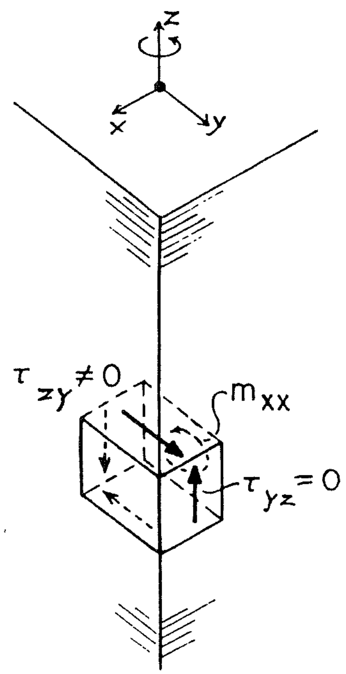

Fig. 5. Free-body diagram of a corner element of a bar in torsion, made of a continuum which supports couple stresses.

rubber cement. A line can also be drawn with ink as was done with the rubber bar, but the line tends to be rather broad, due to the structure of the foam. The bars in figures 3 and 4 were 10 mm by 25 mm in cross section and the cell size of the foam was about 1 mm. Larger bars of foam 50 mm by 50 mm in cross section have also been used successfully for this demonstration. For comparison, rubber bars of comparable size were cast from silicone rubber caulk.

The behavior of the crack in the corner of the foam bar is at variance with the predictions of the theory of elasticity and of the elementary argument presented above and often presented in introductory courses in the mechanics of materials. This may be explained by observing that the foam is actually a structure, not a continuum. The structural elements transmit moments as well as forces, so the free-body diagram of the corner element differs from that of the elastic continuum. The corner element of the structure experiences nonzero loads upon the constituent structural elements, when the bar is twisted. These loads are relieved when a nick is made in the corner, so the surfaces of the nick become displaced.

In introductory courses in the mechanics of deformable bodies, the above demonstration has been used following the analysis of torsion and the discussion of warp of cross sections in members of noncircular cross section. The students are first challenged to explain the corner crack phenomenon, then an

explanation is given, followed by a discussion of composite materials. The demonstration is found to not only be useful pedagogically, but also to excite considerable interest in composites.

The demonstration of the corner crack shear phenomenon has also been used in graduate courses in the theory of elasticity and in oral examinations for the Ph.D. Advanced graduate students are no more successful than sophomores in explaining the behavior of the corner crack.

In courses in elasticity theory, an alternative approach is possible. The free-body diagram of the corner element can be redrawn with the set of moments upon the struts in the structure replaced by a distributed moment per unit area, or couple stress. The ensemble of forces upon the struts is replaced by a force per unit area or the conventional stress. Recall that the differential element was chosen to be a cube of unit side. The corner crack demonstration can then serve to introduce the idea of couple stresses and Cosserat continua in advanced courses in elasticity.

If the idea of couple stress is admitted, the phenomenon of shear of the corner crack can be treated within the context of continuum theory. The matter is interpreted not as a failure of the continuum hypothesis but as a failure of the hypothesis that the interaction across the surface is describable solely in terms of a force vector: a couple vector is also required.

From this point of view, a nonzero state of stress and couple stress is permitted to occur in the corner element. These stresses are then relieved locally when a nick is made, so the faces of the nick undergo relative displacement.

Some writers (e.g. A.P. Boresi and P.O. Lynn, Elasticity in Engineering Mechanics, Prentice Hall, NJ, 1974) prefer to introduce the topic of couple stresses at the senior or first year graduate level, however we have not attempted to do this. For those who favor such an approach, the demonstration of corner crack shear can be of use in developing a physical perception of the idea of couple stress.

In summary, the demonstration of shear at a corner crack is found to be useful in stimulating students to critically examine the assumptions underlying the development of the mechanics of deformable bodies.