Stability of Cosserat solids: size effects, ellipticity and waves

Roderic S. Lakes
Department of Engineering Physics, Engineering Mechanics Program
Department of Materials Science, Rheology Research Center,
University of Wisconsin, 1500 Engineering Drive, Madison, WI 53706-1687
Preprint, Journal of mechanics of materials and structures (JoMMS), 13 (1) 83-91, (2018).
DOI: 10.2140/jomms.2018.13.83

December 6, 2018

Abstract

We consider stability in Cosserat solids. To obtain restrictions on elastic constants based on positive definite strain energy, energy terms are tacitly assumed to be independent. In finite size objects, however, the terms are linked in Cosserat materials. Therefore in contrast to classical solids, stability of Cosserat solids appears to depend on the size and shape of the specimen provided strong ellipticity is satisfied. Stability in the presence of stored energy is possible. Solids with microstructure and stored energy offer potential to facilitate attainment of extreme behavior in the presence of spatial gradients. Snap through buckling in torsion is envisaged by analogy to the axial buckling concept used for composites with negative stiffness inclusions. It is possible to support compressive load in a stable manner but to dissipate energy in the presence of spatial gradients as in torsion or bending.

1 Introduction

Extremely high values of physical properties such as mechanical damping [1] [2], stiffness [3], and piezoelectric sensitivity [4] can be attained in systems at the margins of stability, in bistable systems that undergo hysteresis, and in composites with inclusions of negative stiffness. Thus far, metastable and unstable systems have been understood in the context of classical elasticity. However, many such materials are heterogeneous with a coarse structure size. Such materials can be modeled as Cosserat solids.

Cosserat elasticity [5] [6] incorporates sensitivity to gradients of rotation by virtue of the coupling between rotations and stresses. The constitutive equations for linear isotropic Cosserat elasticity, also called micropolar elasticity [7] in the presence of micro-inertia, are as follows.

$$\sigma_{ij} = 2G\epsilon_{ij} + \lambda\epsilon_{kk}\delta_{ij} + \kappa e_{ijk}(r_k - \phi_k) \tag{1}$$

$$m_{ij} = \alpha\phi_{k,k}\delta_{ij} + \beta\phi_{i,j} + \gamma\phi_{j,i} \tag{2}$$

There are two rotation variables. ϕ_m is the rotation of points, called micro-rotation, and $r_k = \frac{1}{2}e_{klm}u_{m,l}$ is "macro" rotation based on the antisymmetric part of gradient of displacement u_i and e_{jkm} is the permutation symbol. In contrast to classical elasticity the stress σ_{ij} (force per unit area) in Cosserat elasticity can be asymmetric. The theory incorporates a couple stress m_{ij} (a torque per unit area) which balances the distributed moment associated with asymmetric stress.

Isotropic Cosserat solids have six elastic constants, λ, G, α, β, γ, κ. Constants λ and G have the same meaning as in classical elasticity. Cosserat constants α, β, γ provide sensitivity to rotation gradient. The Cosserat constant κ governs the coupling between local rotation and the rotation associated with displacement gradients.

A variety of other generalized continuum theories is available. We use Cosserat elasticity in our laboratory [8] [9] because suitable solutions of boundary value problems are available for the interpretation of experiments.

In the present research, real wave velocity / strong ellipticity are assumed and the size dependence of stability of circular cylinders is explored. The rationale is to provide groundwork for development of gradient sensitive heterogeneous materials with extreme dissipative properties.

2 Range of elastic constants

An assumption of positive definite strain energy leads to restrictions on the elastic constants. For classical isotropic elasticity, the shear modulus $G > 0$ and $3\lambda + 2G > 0$. For Cosserat solids, additional restrictions apply [7] [10]: $\kappa > 0$; $\gamma > 0$; $-\gamma < \beta < \gamma$; $3\alpha + \beta + \gamma > 0$.

As for physical interpretation, positive definite strain energy corresponds in classical elasticity to positive shear modulus and positive bulk modulus. Positive moduli are required for a sample of material to be stable in the absence of constraint.

An assumption of strong ellipticity corresponds to a weaker set of conditions. For classical elasticity they are $G > 0$; $\lambda + 2G > 0$.

Strong ellipticity is associated with the requirement that waves have a real velocity. For longitudinal waves to have a real velocity, $\lambda + 2G > 0$ in both classical and Cosserat elasticity. For transverse waves in Cosserat solids to have a real velocity at low frequency, $G > 0$; in the high frequency limit [7], $G + \kappa/2 > 0$. A second set of transverse waves can exist above a critical frequency. The wave speed v_3 is real if $\gamma > 0$ and $\kappa > 0$; the high frequency limit is $c_4 = \sqrt{\gamma/\rho j}$ in which ρ is density and j is micro-inertia. For such waves to exist at high frequency, $\gamma > 0$. There are also dispersive micro-rotation waves that exist above a critical frequency governed by micro inertia. Such waves can exist if $\alpha + \beta + \gamma > 0$. As for surface waves, their speed is dominated by the shear modulus G; for Cosserat solids there is a contribution from κ. So the stability condition based on plane waves presented in [7] includes $\lambda + 2G > 0$, $G > 0$, $G + \kappa/2 > 0$, $\kappa > 0$, $\alpha + \beta + \gamma > 0$, $\gamma > 0$.

It was suggested [11] [12] that the criteria for real wave velocity in a Cosserat solid are $\lambda + 2G > 0$, $G > 0$, $\kappa > 0$. For strong ellipticity, the usual classical condition $\lambda + 2G > 0$, is supplemented in the Cosserat case [13] with $G + \kappa > 0$, $\gamma > 0$, $\alpha + \beta + \gamma > 0$.

2.1 Stability of classical solids

Negative values of λ (but not too negative) are consistent with stability; this entails a negative Poisson's ratio $\nu = \frac{\lambda}{2(\lambda+G)}$ which is certainly possible [14]. The stability criterion for Poisson's ratio of unconstrained objects is $-1 < \nu < 0.5$.

Suppose $G < 0$; strain energy is not positive definite. An unconstrained block of material is then unstable to shear deformation because the restoring force is reversed. The structural rigidity or spring constant for shear becomes negative. Any small perturbation causes the block to diverge from its initial state. Similarly a negative bulk modulus $B = \lambda + \frac{2}{3}G < 0$ entails a volume change instability of unconstrained blocks.

It is possible for a solid to have a negative bulk modulus, corresponding to a strain energy that is not positive definite. Such a solid may be stabilized by constraint at the boundaries. The material contains stored elastic energy. Effects of constrained negative bulk modulus have been demonstrated in the laboratory [15].

If strong ellipticity fails, then the material exhibits bands of heterogeneous deformation as is the case in solid to solid phase transformations.

In the following we show that for Cosserat solids, materials with non-positive definite strain energy may be stable without constraint if the specimen is sufficiently large in size.

2.2 Stability of Cosserat solids

As with classical solids, a Poisson's ratio within the range $-1 < \nu < 0.5$ is consistent with stability of a Cosserat solid in the absence of constraint via positive definite strain energy. The constant β can be negative provided $|\frac{\beta}{\gamma}| < 1$. The criterion $3\alpha + \beta + \gamma > 0$ allows negative (but not too negative) values of α.

To obtain the restrictions on elastic constants based on positive definite strain energy, it is assumed that modes of deformation are independent. In classical elasticity, volumetric and shear deformation are indeed independent. Such deformations can occur homogeneously without any strain gradient. In Cosserat elasticity it is assumed [7] that strain ϵ_{ij}, rotation difference $r_k - \phi_k$ and rotation gradient $\phi_{i,j}$ can be varied independently. For a particular specimen geometry, however, these variables are coupled via the equilibrium equations and by the boundary conditions for the specimen. Deformation modes that involve a strain gradient are coupled to a rotation gradient. The specifics depend on specimen size and shape as follows.

Consider the torsional structural rigidity ratio Ω. This is the Cosserat rigidity of a round rod of radius r of a Cosserat solid divided by the corresponding classical rigidity. The classical torsional rigidity is $\frac{M}{\theta} = G[\frac{\pi}{2}r^4]$; M is applied moment and θ is angular displacement per length. If elastic constants obey positive definiteness, there are size effects in which slender rods appear stiffer than thicker ones. The rigidity (Figure 1) depends on the characteristic length for torsion, defined as $\ell_t = \sqrt{\frac{\beta+\gamma}{2G}}$.

Suppose $\frac{\beta}{\gamma} < -1$, or equivalently $\beta + \gamma < 0$, which violates positive definite strain energy. Elastic constants λ, G, γ, κ are assumed to be positive. Then $(\ell_t/r)^2$ becomes negative; ℓ_t is imaginary. This is admissible [16], given the form of the energy function. Physically in the present context, negative Cosserat constants corresponding to an imaginary characteristic length entail a destabilizing influence. Then in Eq. 3, the second term provides a negative contribution to the structural rigidity because $(\ell_t/r)^2 < 0$. The structural rigidity Ω remains positive, however, if the rod is sufficiently thick (Figure 1). Observe that one must have $\alpha+\beta+\gamma > 0$ or the Bessel functions have imaginary arguments, which suggests instability. Recall the same criterion was required for micro-rotation waves to exist above a characteristic frequency and to satisfy strong ellipticity. Plots are shown for several values of polar ratio $\Psi = \frac{\beta+\gamma}{\alpha+\beta+\gamma}$. To maintain $\alpha+\beta+\gamma > 0$, $\Psi < 0$ is assumed when $\ell_t^2 < 0$. This corresponds to large positive α. Allowing $\frac{\beta}{\gamma} < -1$ can cause torsional instability only for sufficiently slender rods. Thicker ones are stabilized by the classical term. The stiffening effect does not allow $G < 0$ for slender rods because they would be unstable to deformation in simple shear. The exact solution for torsion of a round rod is [17]:

$$\Omega = (1 + 6\{\frac{\ell_t}{r}\}^2)[\frac{1 - \frac{4}{3}\Psi\chi}{1 - \Psi\chi}], \tag{3}$$

in which $\chi = I_1(pr)/prI_0(pr)$, $p^2 = 2\kappa/(\alpha+\beta+\gamma)$ and I_0 and I_1 are modified Bessel functions of the first kind. Ω is the ratio of torsional rigidity of the Cosserat solid to that of a classical solid.

The case $N = 1$ (in which $N = \sqrt{\frac{\kappa}{2G+\kappa}}$), Eq. 4, is simpler; the same phenomena occur as well. For Cosserat elasticity in the regime $N = 1$, $\frac{M}{\theta} = G[\frac{\pi}{2}r^4](1 + 6(\frac{\ell_t}{r})^2)$. G is the true shear modulus in the absence of gradients.

$$\Omega = 1 + 6(\frac{\ell_t}{r})^2. \tag{4}$$

The case $N = 1$ is called couple stress elasticity. Stability for anisotropic couple stress solids has been analyzed [18, 19, 20]. Strong ellipticity does not imply real wave propagation in such couple stress solids, in contrast to classical solids; similarly for the reverse. Also in the vicinity of the ellipticity criterion, the couple stress solid exhibits folding and faulting patterns not present in the classical case.

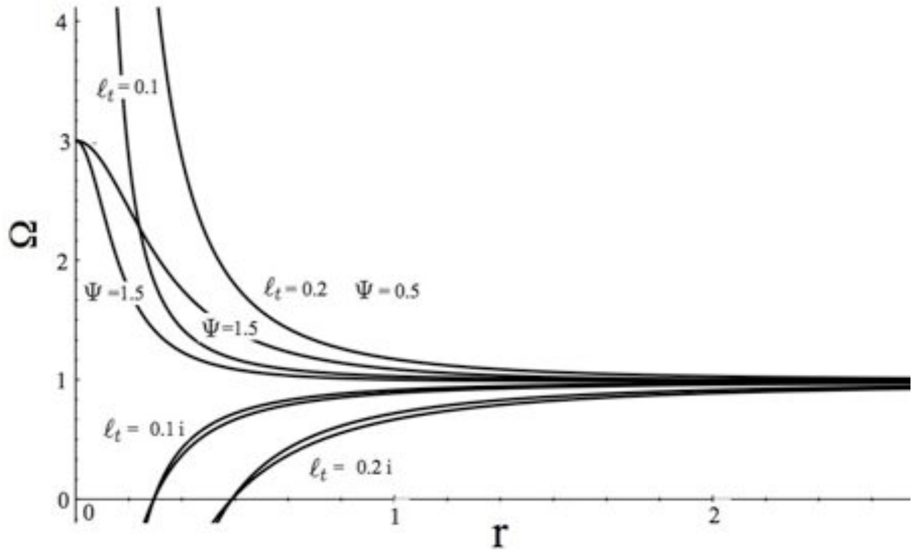

Figure 1: Size effect, rigidity ratio of a round bar vs. radius r, in torsion for $N = 0.5$ and for various ℓ_t^2 including negative values for $\Psi = \pm\, 0.5$ and for $\Psi = \pm\, 1.5$. When the normalized torsion rigidity Ω becomes negative, the bar is unstable in torsion.

We remark that for $\Psi = 1.5$, its upper limit, $\alpha < 0$, assuming positive definite strain energy. For small N, the rigidity softens below classical values but remains positive, as shown in Figure 2, in harmony with the stability criterion of positive definiteness. This region was not explored in [17] and is not known experimentally. The effect of N is not obvious in Eq. 3, however the limit of small N can be visualized via expansion of the Bessel functions for small argument. In the limit, $\chi \to \frac{1}{2}$, then for $\Psi = 1.5$ and small N, the rigidity ratio $\Omega \to 0$. To approach the limit of classical elasticity for a given range of radius, $\alpha, \beta, \gamma \to 0$, which is not plotted in Figure 2.

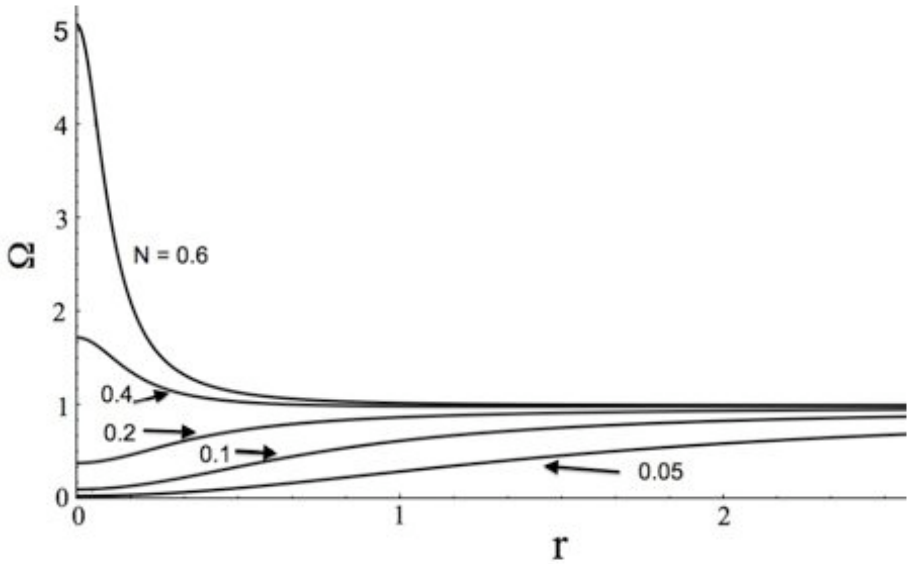

Figure 2: Size effect, rigidity ratio Ω of a round bar vs. radius r, in torsion for $\ell_t = 0.1$ and for $\Psi = 1.5$, the upper limit assuming positive definite energy. Curves are shown for $N = 0.6$, 0.4, 0.2, 0.1, 0.05.

As for bending, size effects in rigidity also occur, Eq. 5, Eq. 6. As with torsion, allowing $\ell_t^2 < 0$, hence $\frac{\beta}{\gamma} < -1$, can cause instability only for sufficiently slender rods (Figure 3). Thicker ones are stabilized by the classical term. The rigidity ratio for bending of a Cosserat elastic circular rod of radius r is [21],

$$\Omega = 1 + 8(\frac{\ell_b}{r})^2 \frac{(1-(\frac{\beta}{\gamma})^2)}{(1+\nu)} + \frac{8N^2}{(1+\nu)}[\frac{(\frac{\beta}{\gamma}+\nu)^2}{\zeta(\delta r) + 8N^2(1-\nu)}] \tag{5}$$

with $\ell_b = \sqrt{\frac{\gamma}{4G}}$ as a characteristic length for bending, $N = \sqrt{\frac{\kappa}{2G+\kappa}}$ as the coupling number, and ν as Poisson's ratio. Also $\delta = N/\ell_b$ and $\zeta(\delta r) = (\delta r)^2[((\delta r)I_0((\delta r)) - I_1((\delta r)))/((\delta r)I_0(\delta r) - 2I_1(\delta r))]$. The classical bending rigidity is $\frac{M}{\theta} = E[\frac{\pi}{4}r^4]$.

The Bessel term vanishes for $\frac{\beta}{\gamma} = -\nu$ and is small for $N << 1$. In this regime we have the following simpler form. This form also applies for $(\ell_b/r)^2 << 1$ but is not as good an approximation.

$$\Omega = 1 + 8(\frac{\ell_b}{r})^2 \frac{(1-(\frac{\beta}{\gamma})^2)}{(1+\nu)}. \tag{6}$$

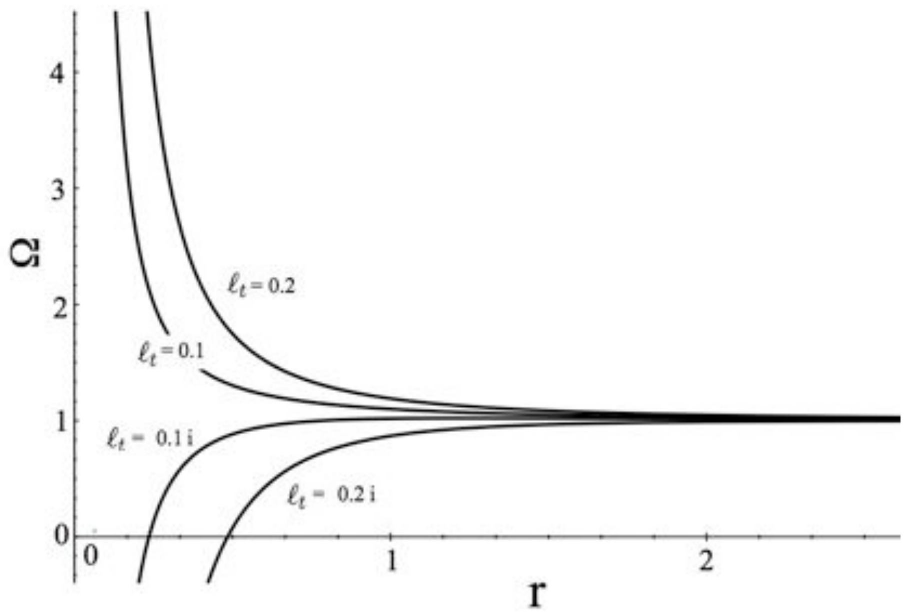

Figure 3: Size effect, rigidity ratio Ω of a round bar vs. radius r, in bending for $N = 0.5$, for $\ell_b = 0.2$ and for various ℓ_t^2 including negative values. When the normalized bending rigidity Ω becomes negative, the bar is unstable in bending.

As for other instability modes, the bar is stable under perturbations in extension because Young's modulus E is positive and there are no rotation gradients. The rigidity depends only on E and not on any Cosserat constants. Similarly the bar is stable under perturbations in uniform shear because the shear modulus G is positive and simple shear is also spatially uniform. Similarly a positive bulk modulus B provides stability with respect to homogeneous volumetric deformation. Sausage deformation involves shear but the gradients in rotation are small compared to gradients in torsion and bending so such deformation is unlikely to cause instability. Also, sausage deformation could initiate from an instability in surface wave speed. However this wave speed depends primarily on G and has a correction involving κ with no effect of β or γ, so the cases considered are stable with respect to surface waves. We do not, however, know *all* possible instability modes.

Observe that the threshold radius for instability is similar in torsion and in bending. In contrast to classical solids, stability of unconstrained Cosserat solids appears to depend on the size and shape of the specimen.

3 Experiment

Stored energy may be achieved via an initial compressive strain, sufficient to buckle some of the cell ribs. Such a deformation was input to a square cross section bar of foam of dimensions 50 x 50 x 161 mm. An open cell polyurethane foam with cells about 4 mm across and density 0.026 g/cc was used. Cyclic sinusoidal deformation at 1 Hz in torsion was applied using a tension-torsion frame with a sensitive torque cell. Calculated peak shear engineering strain was $\pm$ 20%. Torsion hysteresis was measured with and without pre-compression.

Axial pre-compression was found to result in softening of the apparent torsion rigidity and increase in the damping tan δ. For no pre-compression, tan $\delta = 0.12$. For 5% pre-compression, the typical threshold for rib buckling, tan $\delta = 0.11$, and the effective shear modulus softened to 85% of the value for no compression. For 7.5% pre-compression, tan $\delta = 0.19$, and the effective shear modulus softened to 77% of the value for no compression. For 10% pre-compression, tan $\delta = 0.21$, and the effective shear modulus softened to 74% of the value for no compression.

Resolution was insufficient to reveal hypothetical regions of local negative structural rigidity. Rib buckling is known to occur, leading to multiple snap through events. Softening of moduli and increase in damping due to such mechanisms are known in other contexts [2].

The experiment reveals coupling between axial pre-compression and increase in torsional damping and softening of torsional rigidity. Ordinarily in cellular solids the damping tan δ is independent of the foam cell geometry. If the frequency is sufficiently low that stress induced air flow does not contribute to the damping, foam damping is equal to that of the polymer from which it is made. The damping increase in the present experiment occurs when the threshold for cell rib buckling is exceeded, so the increase is interpreted in the context of instability. However to explicitly demonstrate a Cosserat aspect of these phenomena, a protocol to measure size effects with and without pre-compression would require higher resolution than is currently available.

4 Composites and applications

Composite materials with negative stiffness inclusions are well known [1]; they allow extreme values of mechanical damping or of stiffness as shown by experiment [2] [3]. In distributed composites, negative stiffness is achieved by incipient phase transformation of inclusions. In lumped or discrete systems, negative stiffness is achieved by post-buckling of structural elements. The negative stiffness, which involves relaxation of the requirement of positive definite strain energy, is stabilized by constraint within a composite or a structure. Thus far the approach [1] [2] [3] has been classical.

Non-positive definite strain energy entails stored elastic energy in the material. If the material is a Cosserat solid, energy may be stored in the rotational degrees of freedom in an unconstrained object. It is possible to store energy in the material by controlled micro-buckling of ribs as has been done in foams and composites. Stored energy may also be incorporated via a pre-strain or during the fabrication of the composite.

As for purely rotational freedom, we envisage torsional snap through elements (Figure 4) as inclusions. Such a discrete element, either used alone with rotational constraint or embedded in a classical matrix, can be stable while containing stored energy in a neutral position. Torsional snap through elements contribute a negative torsion rigidity; in a composite, the total rigidity can be positive by virtue of the stiffness of the surrounding matrix material. Torsion through the element's negative stiffness region can give rise to partial softening of the torsion rigidity; hysteresis, and mechanical damping, corresponding to similar phenomena observed in classical composites with negative stiffness inclusions. Consequently, there is the potential to support compressive load in a stable manner but to dissipate energy in the presence of spatial gradients as in torsion or bending.

Composites based on Cosserat solids outside the usual range of elastic constants offer the potential of extreme values of damping of torsional waves independently of the ability to support axial load. In the context of waves, material properties can approach the margins of strong ellipticity. Wave cut off may be manipulated by control of gradient sensitivity rather than by adding extra inertia as is done in most "meta-materials".

Figure 4: Torsional snap through element with negative incremental torsional rigidity. Left, top view; right, perspective view.

5 Discussion

Stability of materials has been studied in the context of positive definite strain energy and of ellipticity. If a classical material has positive definite strain energy then it is stable in the absence of constraint. If it obeys strong ellipticity then it is stable with respect to the formation of bands of heterogeneous deformation and with respect to instability associated with waves.

More restrictive views than the assumption of positive definiteness have been taken. In classical elasticity, it had been believed, and even stated in textbooks, that Poisson's ratio of isotropic solids must be positive; negative values are now well known [14]. In Cosserat solids it had been claimed that $\kappa/2 < G$ but this claim was shown to be incorrect [22]. Also, allowing the full stable range of Cosserat constants implies torsion and bending stiffness can diverge as rod radius becomes small. This has been considered to be mathematically unpalatable and also can present difficulties in obtaining stable fits of experimental data to analytical curves [23]. If, however, $\Psi = 1.5$ and $\beta = \gamma$, then the structural rigidity is bounded as radius becomes small. Size effect experiments on isotropic dense foam [8] are consistent with such a notion. However a definitive experimental test of the concept has not been done; some known Cosserat solids such as bone are highly anisotropic; open cell foams and lattices can exhibit various degrees of anisotropy as well as strong size effects.

The value of $\frac{\beta}{\gamma}$ (pertinent to stability limits) can be probed as follows. For square section bars the cross section shape changes in bending as shown both theoretically and experimentally. Classical linear tilt of lateral surfaces becomes a sigmoid shape in Cosserat solids [24]. Experiment shows for an open cell foam, $\frac{\beta}{\gamma} > 0$ and is consistent with $\frac{\beta}{\gamma} = 1$. If $\frac{\beta}{\gamma}$ were to be sufficiently negative, the curve of lateral surfaces would be of opposite sign.

By contrast, views less restrictive than positive definiteness have been taken. For example, continuum models were compared with corresponding structural lattice analyses. It was argued, based on such comparison, that stability conditions should be applied to volume elements considerably larger than the structural length scale of the material to be studied [25]. This concept leads to stability conditions weaker than those obtained from positive definite strain energy in Cosserat solids. However the role of boundary conditions in objects of finite size was not considered, in contrast to the perspective taken here.

As for experiment, torsional damping increases when axial pre-compression suffices to buckle ribs. This could be useful in vibration damping applications but is not definitive in a Cosserat interpretation. As for waves, transmitted longitudinal wave amplitude in a foam is reduced by compression below and near the cut off frequency [26].

6 Conclusion

In contrast to classical solids, stability of Cosserat solids appears to depend on the size and shape of the specimen provided strong ellipticity is satisfied. For Cosserat solids, materials with non-positive definite strain energy due to stored energy in the rotational freedom, may be stable without constraint if the specimen is sufficiently large in size.

7 Acknowledgements

We gratefully acknowledge support of this research by the National Science Foundation via Grant CMMI-1361832 and by ARO U.S. Army Research Office under Grant W911NF-13-1-0484. We thank Professor R. Vanderby for the use of the tension torsion device and Ron and Jeff for effort with the experiments.

References

[1] R. S. Lakes, Extreme damping in composite materials with a negative stiffness phase, Physical Review Letters, 86, 2897-2900, (2001).

[2] R. S. Lakes, Lee, T., Bersie, A., and Wang, Y. C., Extreme damping in composite materials with negative stiffness inclusions, Nature, 410, 565-567, (2001).

[3] T. Jaglinski, Stone, D. S., Kochmann, D, and Lakes, R. S., Materials with viscoelastic stiffness greater than diamond, Science, 315, 620-622, (2007).

[4] Y. C. Wang and Lakes, R. S., Extreme thermal expansion, piezoelectricity, and other coupled field properties in composites with a negative stiffness phase, Journal of Applied Physics, 90, 6458-6465, (2001).

[5] E. Cosserat and Cosserat, F., *Theorie des Corps Deformables*, Hermann et Fils, Paris (1909).

[6] R. D. Mindlin, Stress functions for a Cosserat continuum, Int. J. Solids Structures, 1, 265-271 (1965).

[7] A. C. Eringen, 1968, Theory of micropolar elasticity. In *Fracture*,1, 621-729 (edited by H. Liebowitz), Academic Press, NY.

[8] R. S. Lakes, Experimental microelasticity of two porous solids, Int. J. Solids and Structures, 22, 55-63 (1986).

[9] Z. Rueger and Lakes, R. S., Experimental Cosserat elasticity in open cell polymer foam, Philosophical Magazine, 96 (2), 93-111, (2016).

[10] A. C. Eringen. Microcontinuum field theories. Springer-Verlag, New York, 1999.

[11] P Neff, A Madeo, G Barbagallo, M V. d'Agostino, R Abreu and I-D Ghiba, Real wave propagation in the isotropic relaxed micromorphic model, INSA-Lyon, 2016.

[12] P Neff, A Madeo, G Barbagallo and M Valerio d'Agostino, R Abreu and I D Ghiba, Real wave propagation in the isotropic relaxed micromorphic model, arXiv:1605.07902v1 [math-ph] (2016); Proc. Royal Soc. London 473 (2197), 2017 doi = 10.1098/rspa.2016.0790.

[13] J Jeong and P Neff, Existence, Uniqueness and Stability in Linear Cosserat Elasticity for Weakest Curvature Conditions, Mathematics and Mechanics of Solids, 15: 78-95, 2010.

[14] R. S. Lakes, Foam structures with a negative Poisson's ratio, Science, 235 1038-1040 (1987).

[15] B. Moore, Jaglinski, T., Stone, D. S., and Lakes, R. S., Negative incremental bulk modulus in foams, Philosophical Magazine Letters, 86, 651-659, (2006).

[16] R. D. Mindlin (1965). Second gradient of strain and surface-tension in linear elasticity. International Journal of Solids and Structures 1, 417-438.

[17] R. D. Gauthier and W. E. Jahsman, A quest for micropolar elastic constants. J. Applied Mechanics, 42, 369-374 (1975).

[18] P. A. Gourgiotis and Bigoni, D. (2016). Stress channelling in extreme couple-stress materials Part I: Strong ellipticity, wave propagation, ellipticity, and discontinuity relations. Journal of the Mechanics and Physics of Solids 88: 150-168.

[19] P. A. Gourgiotis, and Bigoni, D. (2016). Stress channelling in extreme couple-stress materials Part II: Localized folding vs faulting of a continuum in single and cross geometries. Journal of the Mechanics and Physics of Solids 88: 169-185.

[20] D. Bigoni and Gourgiotis, P.A. (2016). Folding and faulting of an elastic continuum. Proceedings of the Royal Society A: Mathematical, Physical and Engineering Science 472(2187): 20160018.

[21] G. V. Krishna Reddyand Venkatasubramanian, N. K., On the flexural rigidity of a micropolar elastic circular cylinder, J. Applied Mechanics 45, 429-431 (1978).

[22] S C Cowin, An incorrect inequality in micropolar elasticity theory. Zeitschrift für angewandte Mathematik und Physik ZAMP, 21, 3, 494-497 May 1970

[23] P Neff, J Jeong, A Fischle, Stable identification of linear isotropic Cosserat parameters: bounded stiffness in bending and torsion implies conformal invariance of curvature, Acta Mech 211, 237-249 (2010)

[24] R. S. Lakes and Drugan, W. J., Bending of a Cosserat elastic bar of square cross section - theory and experiment, Journal of Applied Mechanics, 82(9), 091002 (2015). (8 pages).

[25] O. Brulin and Hjalmars, S., Stability conditions in continuum models of discrete systems, in Continuum Models of Discrete Systems 4, eds. O. Brulin and R. K. T. Hsieh, p. 209-212, North Holland Publishing, Amsterdam, 1981.

[26] S. Balch and Lakes, R. S., Amelioration of waves and micro-vibrations by micro-buckling in open celled foam, Cellular Polymers, 36, 1-11, (2017).