Physical meaning of elastic constants in Cosserat, void, and microstretch elasticity

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Abstract

The physical meaning of Cosserat, void, and microstretch elastic constants is analyzed and interpreted. Various torsion experiment designs provide a clear path to extract Cosserat elastic constants independently of any dilatation gradient sensitivity the material may have. For void elasticity (with sensitivity to dilatation gradients) there is no known quasi-static modality to demonstrate phenomena or extract elastic constants independently of any sensitivity to rotation gradients. Wave methods may be appropriate if there is minimal viscoelastic dispersion. Microstretch elasticity, which includes sensitivity to gradients of rotation and of dilatation could account for bending effects larger than those of Cosserat elasticity.

1 Introduction

The amount of freedom embodied in a theory of elasticity is not imposed by the requirement of mathematical consistency. For example, the early uniconstant elasticity theory of Navier [1] has only one elastic constant and Poisson's ratio must be $\frac{1}{4}$ for all materials. This theory is based upon the assumption that forces act along the lines joining pairs of atoms and are proportional to changes in distance between them. This theory was abandoned based on *experiment* that showed materials exhibit various values of Poisson's ratio. Classical elasticity has two independent elastic constants for isotropic materials; the Poisson's ratio can have values between -1 and 0.5. Cosserat elasticity [2] [3] (micropolar [4] if one incorporates an inertia term) has more freedom than classical: points can rotate as well as translate; an isotropic material has six elastic constants. The microstructure elasticity theory of Mindlin [5], also called micromorphic elasticity, has even more freedom; it allows points in the continuum to translate, rotate, and deform. This adds considerable complexity; for an isotropic micromorphic solid, there are 18 elastic constants. One can also incorporate a local dilatation variable [6] as done in void elasticity (5 elastic constants) or combine a dilatation variable with Cosserat elasticity as in microstretch elasticity [7] (9 elastic constants). In nonlocal elasticity, stress at a point depends explicitly on strain in a region around that point [8]. Nonlocal integrals have been approximated as a differential form [9] that entails a simple sensitivity to strain gradients; such gradient approximations have been used and called nonlocal.

The rationale for using a generalized continuum theory with complexity greater than that in classical elasticity is to enable one to interpret phenomena associated with nonzero size of material microstructure and to correctly make predictions of stress and strain fields when gradients of stress or strain are present. There is no length scale in classical elasticity. Length scales do occur in the definition of fracture toughness. Also, toughness of foams is related to the size scale of the cells in the foam [10]. Stress concentrations such as holes and cracks entail large gradients of strain. Generalized continuum theories predict stress concentration factors that differ from those of classical elasticity. Understanding the robustness of heterogeneous materials can benefit from a generalized continuum approach. In composite materials and in biological materials, structural length scales may be non-negligible in comparison with length scales associated with heterogeneity of stress around stress concentrations or in experiments in torsion or bending or indentation. In any material at the nano-scale, length scales in the material's fine structure are likely to be non-negligible in comparison with specimen size. In such cases, classical elasticity is unlikely to suffice for adequate predictions.

In a review of experimental methods for Cosserat solids [11]; Cosserat elasticity was compared with other generalized continuum theories, especially nonlocal elasticity; microstretch elasticity was not considered. More recently, microstretch elasticity has become a topic of interest. Also, our laboratory has recently observed evidence of strong Cosserat effects in reticulated foam [12] and in negative Poisson's ratio foam [13]. In this article, elastic constants in several generalized continuum theories are analyzed and elucidated. First the theories are presented, compared and discussed. Predictions of behavior are presented with interpretation of elastic constants. Prior experiments are reviewed and guide for future experiments is provided.

2 Cosserat elasticity

Of the generalized continuum theories, Cosserat elasticity has been the most thoroughly studied. The Cosserat theory of elasticity [2] [3] incorporates a local rotation of points as well as the translation of points in classical elasticity. In addition to the stress (force per unit area) in classical elasticity, Cosserat elasticity incorporates a couple stress (a torque per unit area). Eringen [4] added a local inertia term called micro-inertia and renamed Cosserat elasticity micropolar elasticity. At frequencies low enough that local resonances are not approached, Cosserat and micropolar are used interchangeably.

The physical origin of the Cosserat couple stress is the summation of bending and twisting moments transmitted by structural elements in materials. The local rotation corresponds to the rotation of structural elements.

The Cosserat theory of elasticity is a *continuum* theory that entails a type of nonlocal interaction. The stress σ_{ij} (force per unit area) can be asymmetric. The distributed moment from this asymmetry is balanced by a couple stress m_{ij} (a torque per unit area). Cosserat elasticity incorporates a microrotation vector ϕ_i that is kinematically distinct from the macrorotation vector $r_i = (e_{ijk}u_{k,j})/2$. ϕ_i refers to the rotation of points, while r_i refers to the rotation associated with translation motion of nearby points; e_{ijk} is the permutation symbol; $\epsilon_{ij} = (u_{i,j} + u_{j,i})/2$ is the small strain tensor. The antisymmetric part of the stress is related to rotations. $\sigma_{ij}^{antisym} = \kappa e_{ijm}(r_m - \phi_m)$ in which κ is an elastic constant. The constitutive equations [4] for linear isotropic Cosserat elasticity are:

$$\sigma_{ij} = 2G\epsilon_{ij} + \lambda\epsilon_{kk}\delta_{ij} + \kappa e_{ijk}(r_k - \phi_k) \tag{1}$$

$$m_{ij} = \alpha\phi_{k,k}\delta_{ij} + \beta\phi_{i,j} + \gamma\phi_{j,i} \tag{2}$$

λ and G are classical elastic moduli. The physical meaning of G is clear; it is the shear modulus used in engineering and represents a material stiffness for shear deformation. The physical meaning of λ is $\lambda = C_{12}$, the elastic modulus component [14] which couples strain in one direction with stress in an orthogonal direction, with all other strains held constant. Further, $\lambda = B - \frac{2}{3}G$, with B as the bulk modulus. The remaining four of the six isotropic Cosserat elastic constants are α, β, γ which provide sensitivity to rotation gradient; κ is a modulus that quantifies the degree of coupling between macro and micro rotation fields. The following technical elastic constants [15] facilitate physical interpretation.

$$Young's\ modulus \qquad E = \frac{G(3\lambda + 2G)}{\lambda + G} \tag{3}$$

$$Shear\ modulus \qquad G \tag{4}$$

$$Poisson's\ ratio \qquad \nu = \frac{\lambda}{2(\lambda + G)} \tag{5}$$

$$Characteristic\ length,\ torsion \qquad \ell_t = \sqrt{\frac{\beta + \gamma}{2G}} \tag{6}$$

$$Characteristic\ length,\ bending \qquad \ell_b = \sqrt{\frac{\gamma}{4G}} \tag{7}$$

$$Coupling\ number \qquad N = \sqrt{\frac{\kappa}{2G + \kappa}} \tag{8}$$

$$Polar\ ratio \qquad \Psi = \frac{\beta + \gamma}{\alpha + \beta + \gamma} \tag{9}$$

The Young's modulus, shear modulus, and Poisson's ratio are the values observable in experiments that impose uniform stress without any gradient [15]. The characteristic lengths appear in analyses of torsion and bending [15]; structural rigidity increases as the thickness of a rod or plate assume a sufficiently small multiple of the characteristic length. The coupling number N is a dimensionless measure of the degree of coupling between the rotation and displacement fields. The limit $N = 1$ corresponds to "couple stress elasticity". The dimensionless polar ratio (of rotation sensitivity moduli) is analogous to Poisson's ratio in classical elasticity. We remark that a parameter $\delta = \frac{N}{\ell_b}$ was used in an analysis of bending of a circular cylinder [16] of a Cosserat solid. Further, Eringen [4] used $\mu + \kappa/2$ to represent the shear modulus G observed in the absence of gradients. Because that causes μ to differ from the observed shear modulus contrary to the usual interpretation, we do not use that notation.

Limits on elastic constants imposed by energy considerations [15] are $G > 0$, $B > 0$, $\ell_b > 0$, $\ell_t > 0$, $0 < N < 1$, $0 < \Psi < 3/2$, $-1 < \frac{\beta}{\gamma} < 1$.

3 Microstretch elasticity

The microstructure elasticity theory of Mindlin [5] allows points to deform as well as to translate as in classical elasticity and to rotate as in Cosserat elasticity. The theory allows a continuum interpretation of wave dispersion observed in crystal lattices. The points in the continuum may be envisaged as local unit cells of the physical microstructure. A relative deformation, the difference between gradient of macroscopic displacement and micro-deformation is defined. Strain energy arguments couple relative deformation to relative stress. The gradient of micro-deformation is coupled to double stress which is a ratio of pairs of forces per area. Double stress includes the Cosserat couple stress. Microstructure elasticity requires 18 elastic constants for an isotropic material. The complexity of this theory presents challenges for interpretation.

Microstretch elasticity [7] is a subset of Mindlin microstructure [5] / micromorphic elasticity. Microstretch elasticity is a generalized continuum representation that includes the rotation sensitivity of Cosserat elasticity and the sensitivity to dilatation gradient associated with void theory (§4). Microstretch elasticity was stated to be simpler than microstructure elasticity; indeed, it entails 9 static isotropic elastic constants rather than 18. The constitutive equations for isotropic material at constant temperature are:

$$\sigma_{ij} = 2G\epsilon_{ij} + (\lambda u_{k,k} + \lambda_0\phi)\delta_{ij} + \kappa e_{ijk}(r_k - \phi_k) \tag{10}$$

$$m_{ij} = \alpha\phi_{k,k}\delta_{ij} + \beta\phi_{i,j} + \gamma\phi_{j,i} \tag{11}$$

$$3s = \lambda_1\phi + \lambda_0 u_{k,k} \tag{12}$$

$$\lambda_k = a_0\phi_{,k} \tag{13}$$

Here λ_k is called [7] an internal traction that causes local dilatation; it is also called [6] an equilibrated stress vector; λ_k is the dilatational component of double stress [5] which Eringen calls first stress moments. Double stress refers to pairs of force per unit area; if the pair of forces generates a moment, then it is a couple stress. Variable s is the difference between the trace of a local stress and the trace of the true stress; it is also called [6] intrinsic equilibrated body force; u is displacement and ϕ represents a local dilatation variable, a change [6] in volume fraction. Observe that ϕ as a scalar has no relation to the local rotation vector ϕ_k in Cosserat elasticity.

Nonclassical contributions to the stress σ_{ij} include (i) an asymmetric contribution due to Cosserat rotation difference coupled via Cosserat κ and (ii) a hydrostatic contribution due to local dilatation ϕ coupled via the microstretch constant λ_0.

As for elastic constants, λ and G are classical elastic moduli, α, β, γ are Cosserat elastic constants that provide sensitivity to rotation gradient, and κ is a Cosserat elastic constant that quantifies the coupling between macro and micro rotation fields. New microstretch elastic constants include a_0, λ_0, and λ_1. Constant a_0 provides sensitivity to gradient of local dilatation, λ_0 is a microstretch modulus that couples dilatation variable change to stress and dilatation to equilibrated body force s, and λ_1 is a modulus that couples local dilatation variable ϕ to s.

As with Cosserat elasticity, characteristic lengths may be defined for microstretch elasticity. The most natural definition is in terms of a ratio of a non-classical elastic constant to a classical one. So, we define a characteristic length in terms of the ratio of the gradient sensitivity constant to a classical modulus; C_{1111} is the constrained modulus tensor element.

$$\ell_m = \sqrt{\frac{3a_0}{C_{1111}}} \tag{14}$$

In some analyses, more complex length parameters arise. For example in a study of bending [17] a variable ζ was defined with dimensions of inverse length.

$$\zeta = \sqrt{\frac{1}{3a_0}(\lambda_1 - \frac{\lambda_0^2}{\lambda + 2G})} \tag{15}$$

But $\lambda + 2G = C_{1111}$, the constrained modulus. So by analogy to Cosserat elasticity we define a coupling coefficient in terms of a ratio of generalized continuum moduli to a classical modulus.

$$N_m = \sqrt{\frac{\lambda_1 - \frac{\lambda_0^2}{\lambda+2G}}{C_{1111}}} = \sqrt{\frac{\lambda_1}{C_{1111}} - \frac{\lambda_0^2}{C_{1111}^2}} \tag{16}$$

So $\zeta = N_m/\ell_m$. So size effects in bending are influenced by the coupling parameter; this is analogous to a parameter $\delta = \frac{N}{\ell_b}$ that was used in an analysis of bending of a circular Cosserat cylinder [16]. In contrast to Cosserat elasticity, the observed value of C_{1111} even under uniform strain will be influenced by the dilatational degrees of freedom [17] [6].

It is also expedient for simplification of bending size effect analysis to define

$$N_{m0} = \frac{\lambda_0}{C_{1111}} \tag{17}$$

$$N_{m1} = \frac{\lambda_1}{C_{1111}} \tag{18}$$

So $N_m = \sqrt{N_{m1} - N_{m0}^2}$.

Energy based limits on the elastic constants [7] [17] include those for classical and Cosserat elasticity as well as $a_0 > 0$, $\lambda_1 > 0$, and

$$3\lambda + 2G - 3\frac{\lambda_0^2}{\lambda_1} > 0 \tag{19}$$

The last expression Eq. 19 may be written $\frac{1}{3}\frac{1+\nu}{1-\nu} > \frac{N_{m0}^2}{N_{m1}}$ or $N_{m1} > 3\frac{1-\nu}{1+\nu}N_{m0}^2$. For Poisson's ratio 0.3, the energy based limit implies $N_{m1} > 1.62 N_{m0}^2$. Because $\lambda = B - \frac{2}{3}G$, Eq. 19 may also be written in terms of the bulk modulus B, $B - \frac{\lambda_0^2}{\lambda_1} > 0$.

4 Void elasticity

Cosserat elasticity incorporates sensitivity to gradients of rotation by virtue of the coupling between rotations and stresses. It is also possible to supplement classical elasticity with sensitivity to gradients of dilatation alone via a generalized continuum theory containing a local dilatation variable suggested to be associated with void volume change [6]. In contrast to Cosserat and microstretch elasticity there is no local rotation variable and no sensitivity to rotation gradient. This theory incorporates three elastic constants in addition to the usual two of classical isotropic elasticity. There is also a rate parameter that is omitted here because viscoelastic behavior can be incorporated in any generalized elasticity theory using the elastic viscoelastic correspondence principle which allows solutions to viscoelasticity problems to be obtained from known solutions of elasticity problems via Laplace or Fourier transforms [18] [19]. The constitutive equations for an isotropic material at constant temperature are:

$$\sigma_{ij} = 2G\epsilon_{ij} + (\lambda\epsilon_{kk} + \beta_v\phi)\delta_{ij} \tag{20}$$

$$h_i = \alpha_v\phi_{,i} \tag{21}$$

$$g = -\xi\phi - \beta_v\epsilon_{kk} \tag{22}$$

Cowin [6] uses α as a void elastic constant but that symbol is already used as a Cosserat constant (Eq. 2), so here it is called α_v. Similarly β_v is used for void theory to distinguish it from the Cosserat constant β. Comparing Eq. 20 and Eq. 10, via $u_{k,k} = \epsilon_{kk}$, the classical constant λ is seen in both; also, $\lambda_0 = \beta_v$. Cosserat type constants are present in microstretch elasticity and absent in void elasticity. Comparing Eq. 22 and Eq. 12, there is a factor -3 difference on the left side. We already have $\lambda_0 = \beta_v$ so the factor -3 cannot be in λ_1; it must be in g. One may presume that microstretch internal traction λ_k is the same as void equilibrated stress vector

h_i; similarly it appears that microstretch equilibrated body force s in Eq. 12 corresponds to void intrinsic equilibrated body force g in Eq. 22. If that were the case, factors of 3 would appear in λ_0 and λ_1 compared with β_v and ξ. That would contradict the correspondence for the elastic constants from Eq. 20 and Eq. 10. So to avoid contradictions in elastic constants, we make the correspondences $3s = -g$ and $\alpha_v = 3a_0$.

An alternate coupling coefficient [20] which may be expressed $N_c = \frac{\beta_v^2}{\xi}\frac{1}{C_{1111}}$ obeys $0 < N_c < 1$; it may also be written $N_c = \frac{N_{m0}^2}{N_{m1}}$.

Aside from the difference in symbols (Table 1), the void theory is equivalent to microstretch elasticity with the Cosserat type constants α, β, γ, κ set to zero. A comparison of symbols for Cosserat elasticity was given in Ref. [21].

Table 1: Symbols used by various authors.

Parameter	Eringen[7]	Cowin[6]	Iesan[17]
Dilatation gradient sensitivity	a_0	$\alpha_v/3$	σ
Micro-stretch modulus	λ_0	β_v	η
Internal modulus	λ_1	ξ	b
Micro-inertia	j	$\frac{2}{3}k$	-

5 Predictions and experiments

5.1 Cosserat elasticity

The Cosserat theory has been explored the most extensively. For example, a size effect is predicted in the torsion [15] and bending [16] of circular cylinders of Cosserat elastic solids. Thin cylinders are more structurally rigid than expected classically. A stiffening-type size effect is also predicted in the bending of plates [15]. No size effect is predicted in tension. By contrast, in classical elastic solids, there is no size effect in torsion or bending; structural rigidity is proportional to the fourth power of the radius. Detailed plots of size effects are available [15] [16] so they are omitted here. Size effects of large magnitude are possible in torsion if $N \to 1$ or if Ψ is well below its upper limit of 1.5.

The stress concentration factor for a circular hole in a thin Cosserat plate is smaller than the classical value, and small holes exhibit less stress concentration than larger ones [22]. By contrast, the classical stress concentration around a hole is independent of hole size. Strain is redistributed in other situations as well. For example, warp in torsion of a bar of rectangular cross section is less in a Cosserat elastic solid than in a classical elastic solid [23]. The deformation in bending is also altered in Cosserat solids: sigmoid curvature of the lateral surfaces of square cross section bars of Cosserat solids is predicted [24]. As for plane waves in a Cosserat solid, shear waves travel faster at higher frequency; longitudinal waves propagate without frequency dependence. There is no prediction of a cut-off frequency for any waves in Cosserat elasticity.

5.2 Void and microstretch elasticity

We consider these theories together because microstretch elasticity is a superposition of void elasticity and Cosserat elasticity. Some predictions are available for void and microstretch theories.

As for tension / compression, analysis of a uniform stress field reveals that the effective constrained modulus is reduced by the presence of the void degree of freedom [6]; specifically

$$C_{1111}^{effective} = C_{1111} - (\frac{\lambda_0^2}{\lambda_1}) = C_{1111}(1 - \frac{N_{m0}^2}{N_{m1}}) \tag{23}$$

The modulus that would be observed in an experiment is $C_{1111}^{effective}$. It is claimed [6] that one can measure some of the additional elastic constants if the experimenter has an independent method to measure the volume change associated with voids. The difficulty with such an approach is that one cannot necessarily assume that the void / local dilatation degree of freedom in the continuum theory corresponds directly to the concentration of physical voids in a laboratory specimen. Indeed, these generalized continuum degrees of freedom were explored in a homogenization analysis [7] [25] of a mass-spring system in the study of waves; there are no voids in a mass-spring system. Tension was also studied analytically in microstretch elasticity [17]. As is the case with Cosserat elasticity, size effects are predicted not to occur in tension. The predicted effective Young's modulus in tension is reduced by the microstretch variables [17] so the classical moduli no longer have the same meaning as in classical or Cosserat elasticity. Similarly the Poisson's ratio is changed by the microstretch variables [17]. This situation is analogous to the use [4] in Cosserat elasticity of $\mu + \kappa/2$ to represent the shear modulus G observed in the absence of gradients. The difficulty with such notation is that μ is used in elasticity theory to represent the observed shear modulus and it no longer represents the shear modulus when used as above. We avoid the potential for confusion by using G to represent the shear modulus in Cosserat elasticity.

Size effects arise due to sensitivity to dilatation gradient [6]. Size effects are predicted in bending but not in torsion. This bending analysis [6] allowed surface tractions on the lateral surfaces, so it does not represent an exact solution for bending. It is not obvious whether the size effect represents stiffening or softening. Stress concentration around a small hole is predicted to be larger than for a large hole [26] and the stress concentration factor is larger than the value predicted by classical elasticity. This is the opposite effect of Cosserat elasticity. Too, the velocity of longitudinal waves increases with frequency but shear waves are not affected. The generalized continuum theory of voids is in contrast to the classic Biot theory [27] which analyzes the stress-induced pressure and flow of fluids between communicating pores in the solid. Void theory does not contain such physics, but it does allow sensitivity to gradients.

Void theory predicts bending size effects [6] for a bar of width h; the result of the approximate analysis is here considerably simplified in terms of a dimensionless ratio J. A length parameter $\ell_0 = \frac{\ell_m}{N_m}$ with $N_m = \sqrt{N_{m1} - N_{m0}^2}$ as above was used the analysis [6].

$$J = \left(\frac{N_{m0}^2}{N_{m1} \; - \; N_{m0}^2}\right)\left(\frac{1-2\nu}{1+\nu}\right)\left(1 - 3\left(\frac{\ell_0}{h}\right)^3\left(\frac{h}{\ell_0} \; - \; \tanh\left(\frac{h}{\ell_0}\right)\right)\right) \tag{24}$$

The rigidity ratio Ω is the ratio of bend rigidity in void elasticity to bend rigidity in classical elasticity.

$$\Omega = \frac{1-J}{1-\frac{1}{2}J(1+\nu)} \tag{25}$$

Stiffening size effects in bending are possible as shown in Fig. 1. Curves are for $\nu = 0.3$, $N_{m0} = 1$; $N_{m1} = 1.65, 2, 3$.

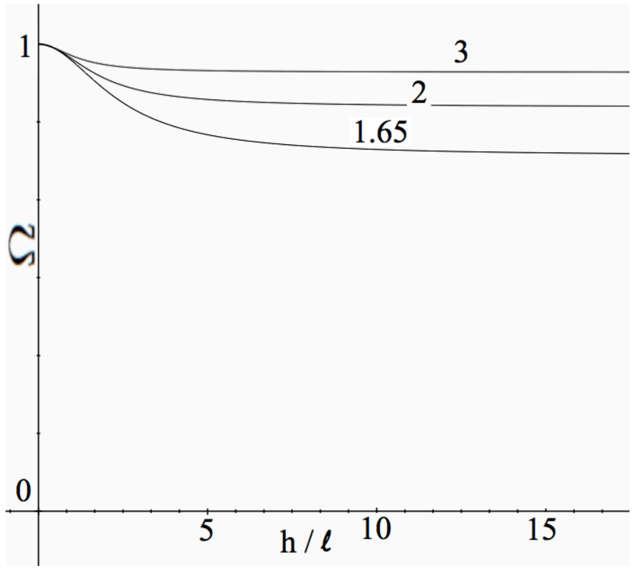

Figure 1: Bending size effect for void theory; normalized rigidity Ω vs. normalized specimen thickness $\frac{h}{\ell_m}$, for $\nu = 0.3$, $N_{m0} = 1$; $N_{m1} = 1.65, 2, 3$.

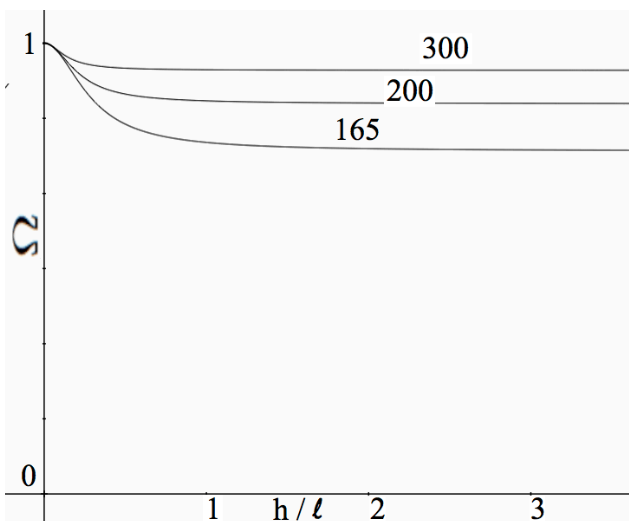

Figure 2: Bending size effect for void theory; normalized rigidity Ω vs. normalized specimen thickness $\frac{h}{\ell_m}$, for $\nu = 0.3$, $N_{m0} = 10$; $N_{m1} = 165, 200, 300$.

Size effects are not of large magnitude in this series, even for elastic constants approaching the stability limit of Eq. 19. This is in contrast with Cosserat elasticity in which large size effects are possible and are observed.

If larger N values are assumed, $N_{m0} = 10$; $N_{m1} = 165, 200, 300$, corresponding to λ_0 larger than C_{1111}, size effects are shown in Fig. 2. Again, size effects are not of large magnitude, in contrast with Cosserat elasticity. All size effects explored within this solution entail increasing apparent modulus as thickness is reduced.

The lowest of the size effect curves corresponds to ratios of elastic constants approaching the stability limit, Eq. 19, but the stiffness does not tend to zero anywhere. The limit from the denominator in Eq. 24 to be positive and nonzero is $N_{m1} > N_{m0}^2$. This is less stringent than the energy limit Eq. 19 for Poisson's ratio 0.3; for Poisson's ratio 0.5 they are equivalent.

The nature of the instability mode must be found elsewhere. The void theory stability limit of Cowin [6] $3\lambda + 2G - 12\frac{\beta_v^2}{\xi} > 0$ differs from that of Eringen [7], Eq. 19 by a factor of four in the last term. The origin of the disparity is unknown.

As for microstretch elasticity, a complicated exact solution for tension and for bending of a circular cylinder was presented [17]; this is much more complicated than the corresponding exact solution [16] for bending in Cosserat elasticity. It is not obvious whether this microstretch size effect represents stiffening or softening.

5.3 Determination of Cosserat elastic constants by experiment

Measurements of torsional size effects is sufficient to demonstrate the presence of Cosserat effects because there is no dilatation or dilatation gradient; void elasticity predicts no size effects in torsion. If the material is a Cosserat solid or a microstretch solid, torsional size effect measurements, if done over a sufficient range in specimen diameter, allow extraction of four of the elastic constants, specifically the shear modulus G, characteristic length in torsion ℓ_t, coupling number N and polar ratio Ψ. One obtains G via a pure shear test without gradient or a torsion test on a sufficiently large specimen or from an asymptote at large diameter in torsion size effect curve. The characteristic length ℓ_t is obtained via size effects in torsion of rods of different radius r for relatively large specimens via the approximate solution for the rigidity ratio

$$\Omega \approx 1 + 6(\ell_t/r)^2 \tag{26}$$

The full Bessel function solution [15] is required to obtain N from size effects on smaller specimens and also Ψ from yet smaller specimens. Such inference is valid for a Cosserat solid or a microstretch solid; there is no dilatation in torsion, hence no effect of dilatation gradients. If the material is a Cosserat solid, Young's modulus E can be obtained from simple tension or compression experiments or from or from an asymptote at large size in a bending size effect curve. Bending of a plate to a cylindrical shape via controlled moments on all edges reveals ℓ_b independently of other constants provided one bends plates of various thickness [15]. Bending of the same circular rod used for torsion can also be used to obtain ℓ_b with confirmation of N via a somewhat more complex procedure using a solution [16] containing Bessel functions. So, for Cosserat solids, measurement of size effects in both bending and torsion suffice to determine all six elastic constants. If the material is a microstretch solid, then bending size effects will contain a Cosserat (rotation gradient) and a dilatation gradient contribution; there are then too many elastic constants to extract them from a single curve.

Cosserat elastic effects have been observed experimentally. Torsion and bending studies on closed cell foams [28], [29] and of compact bone [30] reveal size effects consistent with Cosserat elasticity. The apparent modulus increases substantially as the specimen diameter becomes smaller. This is in contrast to the predictions of classical elasticity. For dense (340 kg/m^3) closed cell polyurethane foam [28], $E = 300$ MPa, $G = 104$ MPa, $\nu = 0.4$, $\ell_t = 0.62$ mm, $\ell_b = 0.33$ mm, $N^2 = 0.04$, $\Psi = 1.5$. The cell size is from 0.05 mm to 0.15 mm. For dense (380 kg/m^3) polymethacrylamide closed cell foam (Rohacell WF300) [29], $E = 637$ MPa, $G = 285$ MPa, $\ell_t = 0.8$ mm, $\ell_b = 0.77$ mm, $N^2 \approx 0.04$, $\Psi = 1.5$. The cell size is about 0.65 mm. The Cosserat characteristic length was also determined in a (two dimensional) polymer honeycomb [31].

Full field measurements of deformation of square section bars in torsion have been conducted to ascertain the predictive ability of Cosserat elasticity. Warp of a bar of rectangular cross section in torsion is predicted to be reduced in a Cosserat elastic solid [23]. The corresponding strain field was observed in compact bone [32]; the non-classical effects observed were in agreement with predictions based on Cosserat constants obtained from size effects observed in separate experiments on circular cylinders. Deformation spills over into the corner of the square section where it would be zero in classical elasticity [33] as revealed by holography. Such strain redistribution ameliorates concentration of strain. Strain at the corner of the section entails asymmetry of the stress as is predicted by Cosserat elasticity. The corresponding reduction of warp deformation of square section bars of dense foam has been observed via holography [34]. The strain field in bending is also altered in Cosserat solids. Sigmoid curvature of the lateral surfaces of bent square cross section bars of open cell foam was observed experimentally and predicted theoretically via Cosserat elasticity [24].

Wave methods could be used for Cosserat solids provided the material has minimal viscoelastic damping. In viscoelastic materials, the dispersion (frequency dependence of velocity) increases with damping. Cosserat elasticity predicts increase of shear wave velocity with frequency [4]; dispersion due to viscoelasticity would be a confounding variable.

5.4 Determination of microstretch elastic constants by experiment

No known experimental results are available for void or microstretch elasticity. However homogenization analysis of a one dimensional mass-spring system was used to calculate the dilatation elastic constants [25] and homogenization analyses of lattices of ribs are available for Cosserat elasticity. In this section the prospects for such experiments are explored. Void elasticity predicts size effects in bending but not in torsion. At first sight such an observation would appear as a clear signature of a void elastic solid, i.e. a microstretch solid with dilatational degrees of freedom but not rotational degrees of freedom. However such an effect could also occur in a Cosserat solid (Eq. 2) if β/γ = -1, the lower limit allowed by energy considerations. If there is reason to infer a larger β/γ, either from the shape of the bending size effect curves or from comparison with other experiments or from homogenization analysis, then a microstretch dilatation gradient contribution is to be suspected. Such size effects do not suffice for the calculation of all the elastic constants of a microstretch solid; there are too many constants to extract from one curve.

Configurations sensitive only to dilatation gradient, not to rotation gradient, are available. For example static pressurization of a hollow cylinder or sphere will generate a gradient in strain without the rotations that would drive Cosserat effects. Solutions are available for the cylindrical and spherical thick wall pressure vessel problems for a material obeying void elasticity [35]. The stress field is not affected by the void degrees of freedom but the displacements are altered giving rise to a larger structural compliance. Curiously the gradient sensitivity parameter α_v hence the characteristic length, does not enter the final solution for the displacements. So the reduction in structural rigidity is similar to that predicted (Eq. 23) for uniform compression, in that there is a change in effective stiffness but no gradient sensitivity parameter α_v. The classical displacement field for the hollow sphere contains r and $1/r^2$ terms with the latter coupled via the shear modulus. The void contribution to displacement contains only an r term; similarly the dilatation variable ϕ is independent of r. This is not promising for experiments based on size effects. In any case, experiments of this or related type are likely to be more difficult than bending or torsion, particularly if the specimen is porous and permeable as with open cell foams.

The presence of dilatation sensitive elastic effects may be studied with waves. As with void theory, microstretch elasticity admits two kinds of longitudinal waves [25]. A cut off frequency phenomenon is predicted: at sufficiently high frequency, the wave speed drops to zero. Viscoelasticity cannot account for such an effect; the effect of viscoelasticity is an increase of wave speed with frequency. Beyond the cut off frequency, no waves propagate. The cut-off angular frequency is $\omega_{cr} = \sqrt{\frac{2\lambda_1}{\rho j}}$ with j as micro-inertia and ρ as density [25]. While it is suggested that measurement of this frequency enables extraction of the elastic constant λ_1, the micro-inertia is not known a priori. It has dimensions of length squared. The constant λ_0 is also needed to interpret size effect studies; the wave cut off frequency has λ_1 but does not reveal λ_0 or a_0 which provides sensitivity to gradient of local dilatation. Experimental results are available for dispersion and cut off frequencies in open cell foam [36]. These experiments were interpreted in the context of structural vibration of cell ribs and also via [5] microstructure elasticity. Cut off frequencies were not sufficient to determine generalized continuum elastic constants. The shape of the longitudinal speed vs. frequency curve might be studied as a way to obtain several constants; the curve depends on four constants, the three dilatational elastic constants and the micro-inertia. Mindlin [5] and Cowin [20] present further detail regarding waves. There are two kinds of longitudinal waves corresponding to acoustic and optical branches observed in crystals. If it is possible to observe both waves, then it might be possible to extract elastic constants. Biot theory [27] also allows two kinds of longitudinal waves. Biot type slow waves have been observed in synthetic [37] and in natural [38] liquid filled permeable materials. Biot slow waves depend on fluid-solid interactions in permeable materials. Waves predicted in void elasticity, by contrast, will occur in a solid material with no fluid in any pores. By removing any fluid from the pores, one may distinguish the origin of the waves. Even so, thus far no experiments with a microstretch interpretation to determine elastic constants are known on physical materials.

6 Conclusions

Cosserat elasticity may be demonstrated and four of the six elastic constants extracted via torsion modalities independently of any dilatation gradient sensitivity the material may have. No comparable quasi-static modality for void elasticity has emerged. In Cosserat solids, methods are known and have been applied to obtain all six elastic constants via quasi-static experiments. Wave methods may be appropriate for both Cosserat and microstretch solids provided viscoelastic dispersion is not too large. Microstretch elasticity, which includes sensitivity to gradients of rotation and of dilatation could account for bending effects larger than those of Cosserat elasticity.

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