IS BONE ELASTICALLY NONCENTROSYMMETRIC?

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Introduction

Generalized continuum theories developed over the last century are thought to be applicable to the description of mechanical behavior in materials with fibrous, granular, and lattice structure. Theories such as Cosserat theory, the indeterminate couple stress theory of Mindlin and Tiersten and the micropolar theory of Eringen all admit couple stresses and an asymmetric force stress tensor. These theories predict stress distributions in loaded objects, which differ from the predictions of classical elasticity; the difference is most marked in regions of high strain gradient, e.g. near a discontinuity or interface. We have discussed elsewhere evidence of non-classical effects in bone. In this abstract, we consider the consequences of lack of a center of symmetry in a material. Bone exhibits piezoelectric-like effects which can be represented by a third-rank tensor, therefore it lacks a center of symmetry.

Theory

We consider the following constitutive equations in micropolar elasticity, which is a generalization of the Mindlin-Tiersten couple stress theory. The material is assumed to be isotropic with respect to coordinate rotations but not with respect to inversions.

$$\sigma_{k\ell} = \lambda e_{rr}\delta_{k\ell} + (2\mu+\kappa)e_{k\ell} + \kappa e_{k\ell m}(r_m-\phi_m) +$$
$$C_1\phi_{r,r}\delta_{k\ell} + C_2\phi_{k,\ell} + C_3\phi_{\ell,k}$$

$$m_{k\ell} = \alpha\phi_{r,r}\delta_{k\ell} + \beta\phi_{k,\ell} + \gamma\phi_{\ell,k} + C_1 e_{rr}\delta_{k\ell} +$$
$$(C_2+C_3)e_{k\ell} + (C_3-C_2)e_{k\ell m}\,(r_m-\phi_m),$$

in which σ is the stress, m is the couple stress, e is the small strain, λ and μ are the Lamé constants, and $\alpha,\beta,\gamma,\kappa,C_1,C_2,C_3$ are micropolar elastic constants. $r_k = (1/2)e_{k\ell m}U_{m,\ell}$ is the macrorotation vector and ϕ_k is the microrotation vector. U is the displacement and $e_{k\ell m}$ is the alternating symbol. If the material is centrosymmetric, the C's vanish and Eringen's original version of the micropolar theory is recovered. If α,β,γ, and κ are zero, the material is classically elastic. The quantity $[(\beta+\gamma)/(2\mu+\kappa)]^{1/2}$ has dimensions of length and is a characteristic length for torsional deformation. For bone this length is comparable to the size of osteons.

If a circular cylinder of radius a, of a noncentrosymmetric micropolar solid is subjected to an axial tensile load and is not subject to rotational constraint, it will undergo torsional as well as axial deformation. The twist angle per unit length is given approximately by:

$$\Phi = \frac{C_1(1-2\nu) + C_2 + C_3}{(\alpha+\beta+\gamma) + (2\mu+\kappa)(a/2)^2}\, e$$

in which ν is Poisson's ratio and e is the axial strain. This relation is approximately true if the characteristic length is small and is exact if $\kappa = 0$. Both Poisson's ratio and the Young's modulus are predicted to exhibit size effects: both become smaller as the specimen diameter becomes smaller. These effects arise from noncentrosymmetry and are not predicted to occur in the symmetric micropolar solid. In both the symmetric and nonsymmetric cases the apparent shear modulus increases with a decrease in specimen size.

Based on energetic considerations we obtain constraints on the C coefficients and therefore on the above nonclassical response to tensile load. For example, $K^2 \equiv [(C_2+C_3)^2/4(2\mu+\kappa)(\beta+\gamma)] \leq 1$. We may consider the nondimensional quantity K^2 as a coupling coefficient in a manner analogous to piezoelectricity theory.

Experiment

In a preliminary test upon a specimen of human compact bone, tensile load has been applied to the specimen by means of a monofilament polymeric line. Angular displacement was measured using a reflected laser beam to eliminate instrumental crosstalk between tension, bending, and torsion. Preliminary results indicate a linear relation between axial strain and angular displacement. The coupling coefficient K^2 is about 15%.

Significance

In the stress analysis of bones and bone-implant systems, the classical theory of elasticity is currently assumed. If bone actually obeys a more generalized theory, then actual stress distributions will differ from those currently predicted. The difference is most marked in regions of high strain gradient, of high stress concentration and near interfaces. Since the interface is considered a critical zone in bone-implant systems, we feel that investigation of the applicabilty of generalized continuum theories to bone to be warranted.

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