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Dynamical Study of Couple Stress Effects in Human Compact Bone

Torsional resonance experiments performed on wet human compact bone disclose effects due to couple stress. The characteristic length, which is an additional material coefficient which appears in couple-stress theory, is of the order of the size of osteons and appears to be smaller at high frequencies than at low frequencies. The presence of couple-stress effects implies a reduction in the stress concentration factor around holes, particularly small holes.

Introduction

Extended Continuum Theories. In the 19th century, Voigt [1, 2] considered the idea that models of a physical body should include effects associated with direction assigned to points within the body, leading to the idea of couple stress. E. and F. Cosserat in 1909 [3] systematically developed a theory of elasticity which included the effects of a couple per unit area upon a material volume as well as of force per unit area [3]. The stress tensor σ in the Cosserat theory is not symmetric; it satisfies the equation

$$1/2(\sigma ij - \sigma ji) = m^{ijk}{}_{,k} \quad (1)$$

in which m^{ijk} is a third rank tensor which represents the couple-stress field [4].

In the couple-stress theory, or theory of asymmetric elasticity, the stress is related not only to the infinitesimal strain, which represents the symmetric part of the displacement gradient tensor, but also to the local rotation, or the antisymmetric part of the displacement gradient. Little was done with these ideas until the early 1950s when similar theories were proposed for anisotropic crystals [5, 6]. Huntington, in a review of work in this area, concluded that in materials without internal electric or magnetic fields, the long-range electromagnetic forces between atoms cannot to first order cause the torques required by such theories [7]. In addition, the foundations of the theories in references [5] and [6], but not those of the Cosserat theory, contain errors or lack of mathematical precision [8].

Generalized elasticity theories of the couple-stress type became subject to renewed interest in the early 1960s: Eringen, Mindlin, Tiersten, and Toupin developed continuum theories which admit an asymmetric stress tensor [9–11]. For example, the following are constitutive equations for a linear, anisotropic solid with couple stress [8]. The usual Einstein summation convention is used:

$$\sigma_{ij}{}^{\text{sym}} = c_{ijkl}\epsilon_{kl} + b_{ijkl}\kappa_{kl}, \text{ and} \quad (2)$$

$$\mu_{ij}{}^{D} = b_{klij}\epsilon_{kl} + a_{ijkl}\kappa_{kl} \quad (3)$$

in which σ^{sym} is the symmetric part of the usual force-stress tensor, ϵ is the strain, μ^D is the deviator of the couple-stress tensor, c is the usual fourth-rank elastic modulus tensor, a and b are material property tensors related to couple stress, and κ is the gradient of the small rotation vector. κ is given by

$$\kappa_{ml} = 1/2\, e_{nkl} u_{k,nm} \quad (4)$$

in which u is the displacement vector, e is the alternating symbol, and the comma denotes differentiation. In terms of the strains, ϵ,

$$\kappa_{ml} = e_{nkl}\epsilon_{mk,n} \quad (5)$$

The antisymmetric part of the stress is given by

$$\sigma_{kl}{}^{A} = 1/2 e_{jlk}\mu^{D}_{ij,i} + 1/2 e_{jlk}\rho\lambda_j + \frac{1}{2} e_{jlk}(\mu_{ii}/3)_{,j} \quad (6)$$

in which ρ is the density and λ is the body torque per unit mass.

In isotropic materials, constitutive equations (2) and (3) become

$$\sigma^{\text{sym}}_{ij} = \lambda\epsilon_{kk}\delta_{ij} + 2\mu\epsilon_{ij} \quad (7)$$

$$\mu^{D}_{ij} = 4\eta\kappa_{ij} + 4\eta'\kappa_{ji} \quad (8)$$

$$l = \sqrt{\eta/\mu} \quad (9)$$

The quantity l has dimensions of length and is referred to as the characteristic length in isotropic couple-stress theory.

In the classical (force-stress) elasticity theory, the coefficients a and b in equations (2) and (3) and also η, η', and l in equation (8) are zero, so that neither local rotations nor strain gradients contribute to the stress. In the couple-stress theory, the stress depends on the strain and on some components of the gradient of the strain. A more complete theory would include all components of the strain gradient

$$\sigma^{\text{sym}}_{ij} = c_{ijkl}\epsilon_{kl} + g_{ijkln}\epsilon_{kl,n} \quad (10)$$

Following these developments, literally hundreds of papers have been published dealing with the solution of specific boundary value problems in couple-stress elasticity. In addition, the theory itself has been further generalized to include nonlinear effects [10, 11], viscoelastic behavior [12], and general nonlocal effects in which the stress at a point is expressed as a functional of the deformation histories of all material points in the solid [13].

Salient consequences of couple-stress theory are as follows:

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(*i*) Calculation of stress concentration factors around a circular hole, taking into account couple stresses, results in lower values than accepted heretofore [14].

(*ii*) Stress concentration around a rigid inclusion in an elastic medium is greater when couple stress is present than when it is not present. The maximum stress occurs at the interface rather than in the medium itself, when couple stress is present. The maximum stress around a soft inclusion is reduced by the presence of couple stress [15].

(*iii*) Dilatational waves propagate nondispersively, i.e., with velocity independent of frequency, in an isotropic elastic medium with couple stress. Shear waves propagate dispersively in the presence of couple stress [9, 16].

(*iv*) A size effect is predicted in the torsion of circular cylinders of elastic materials with couple stress. The effective shear modulus associated with such cylinders increases as their size decreases [17].

(*v*) A similar size effect is also predicted in the bending of plates and of beams [18, 19].

(*vi*) The mode structure of vibrating bodies is modified [9].

Couple-stress theory, like classical elasticity, is a continuum theory which makes no reference to atoms or other structural features of the material which is described. Nevertheless, consideration of structural information for a particular material can lead to a greater understanding of its behavior and to quantitative prediction of its mechanical properties. The physical origin of stress lies in the interatomic forces of attraction and repulsion. Elasticity theory represents more than an analytic description of the phenomenological behavior since it can be derived as a first approximation of the interaction between atoms in a solid. Interatomic forces are short range, but they exert influence farther than one atomic spacing. Therefore, there must be some resistance to lattice curvature [20]. Indeed the very general nonlocal elasticity theory is able to predict the dispersion of elastic waves of wavelength comparable to atomic spacing [21, 22]. Quoting Kröner regarding couple-stress elasticity: "It is therefore no question whether this kind of elasticity exists or not. The question is rather how large the effect might be. Certainly the effect is small in most cases, otherwise one should get remarkably different results in experiments of measuring Young's modulus in a simple tension test first and in a bending test secondly." [20]. One measure of the magnitude of couple-stress effects is the characteristic length l defined as the square root of the ratio of a curvature modulus to an elastic modulus [14]. It might be of the order of the grain size in polycrystalline or granular materials [14].

Explicit calculations of couple-stress coefficients of various model materials with microstructure have been carried out by a number of investigators. For example, in a two-dimensional model composed of orientable points, joined by extensible and flexible rods, the equations obtained in a continuum approximation are identical with those of couple-stress theory [23]. Both the classical elastic constants and the couple-stress coefficients can be expressed in terms of the properties of the classically elastic beams composing the latticework [23]. In another model, a three-dimensional honeycomb structure consisting of thin-walled cubical cells, the model material behaves like a couple-stress solid and the elastic coefficients are calculated from structural considerations [24]. In composites with a laminated structure, couple-stress effects also are predicted to occur and their magnitude is a function of the classical elastic constants of the composite's constituents [25].

Experiments. Experiments on a macroscopic scale upon metal have disclosed no evidence of couple stress. Bending experiments were performed in order to ascertain whether the flexural rigidity, and therefore the effective Young's modulus, would increase with decreasing specimen thickness as predicted by couple-stress theory. The experiments disclosed that the characteristic length l must be less than 0.002 in. (0.05 mm), about the size of the grains in the steel and aluminum used [26], or less than 0.03 mm in aluminum in a different set of tests [27]. Couple stresses, therefore, will not significantly perturb the stress distributions in a macroscopic piece of metal, calculated using classical elasticity [27]. Experiments performed on a model composite also disclosed no evidence of couple stress [28].

Bone Biomechanics. Study of elastic behavior in bone dates to the nineteenth century works of Wertheim and Rauber [29, 30]. The latter investigator demonstrated that bone is anisotropic as well as viscoelastic, but did not explore fully these aspects of its behavior. Many researchers in recent years have determined the Young's modulus E and the shear modulus G for bone under a variety of conditions. Many of these data have been tabulated and discussed in a review by Reilly and Burnstein [31], which presents results prior to 1974.

Bone is anisotropic; that is, its properties are dependent on direction [32]. Therefore, more than two elastic constants are required to describe its behavior. The constitutive equation for an anisotropic solid is

$$\sigma_{ij} = c_{ijkl}\epsilon_{kl}$$

in which c_{ijkl} is the fourth rank elastic modulus tensor and the stress σ is assumed to be symmetric. For the general anisotropic solid, there are 21 independent coefficients in c; for the hexagonal crystal or for the texture symmetry ∞T, there are five, and for the isotropic solid there are two. The five elements of the modulus tensor in bone at ultrasonic frequencies are known [33]. Since bone is viscoelastic, the elements of the modulus tensor may be considered to depend on time or on frequency. The constitutive equation in the linear domain of the behavior is given by

$$\sigma_{ij} = \int_{-\infty}^{t} C_{ijkl}(t-\tau)\frac{d\epsilon_{kl}}{d\tau}\,d\tau$$

Of the five elements of the modulus tensor, only C_{2323} is known over an appreciable domain of time [34].

Although experiments intended to explore couple-stress effects in bone have not been reported in the literature, several reports are suggestive of a failure of classical theory to adequately describe bone. For example, Frasca's study of single osteons and osteon groups discloses size effects [35] which should not occur in a classically elastic solid.

Methods

The experimental technique is aimed toward the determination of effects of specimen size and mode of vibration on the torsional resonance frequency of circular cylinders of bone. Size effects are predicted by couple-stress theory [9], but not by classical elasticity or viscoelasticity. A modified composite piezoelectric oscillator approach is used [36]. The apparatus, shown in Fig. 1, makes use of two torsion mode quartz crystals and a cylindrical specimen cemented end-to-end with a cyanoacrylate adhesive. One crystal, driven electrically by a gated burst sinewave signal, excites torsional vibrations in the system, and the other crystal produces an electrical signal in response to these vibrations. The signal is amplified and highpass filtered, and fed into a storage oscilloscope. In the gated burst approach, the exciting signal is switched on and off periodically and the free decay of vibration is monitored during the off period. This procedure eliminates capacitively coupled interference from the driver to the sensor crystal. Such interference is more of a problem in continuous wave experiments on bone than in studies on

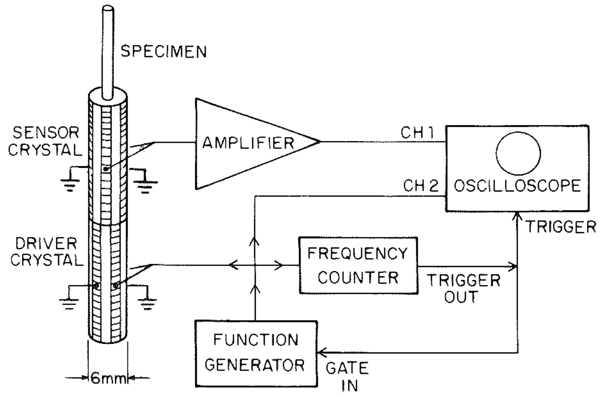

Fig. 1 Experimental apparatus for torsion resonance studies

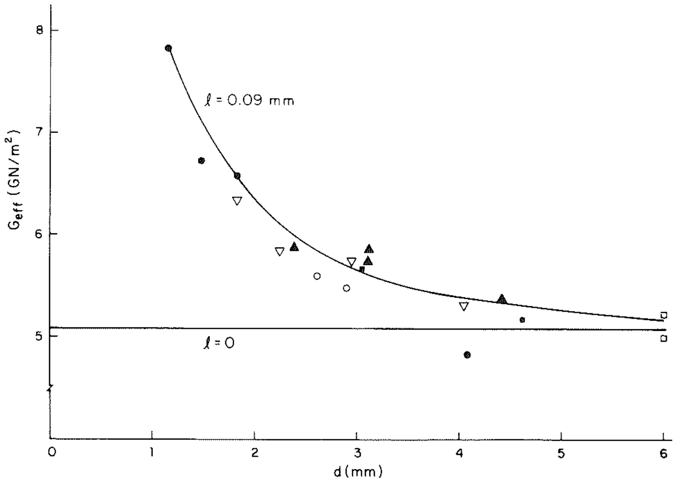

Fig. 2 Observed size effect in the effective shear modulus for wet human compact bone. Curves are based on dynamical theory [9]. Points are experimental. Solid circles are for femoral bone from a male subject 5 ft 9 in., 130 lb, who died at age 27 of hepatic failure. Other symbols represent femoral bone from a male subject 5 ft 10 in., 160 lb, who died at age 57 of Hodgkin's disease. (1 in. = 2.54 cm, 1 ft = 0.3048 m, 1 lb = 0.4536 kg)

metals and ionic crystals reported earlier using the composite oscillator. Bone's loss tangent is comparatively large; therefore its amplitude at resonance for a given level of excitation is small, which results in a reduced ratio of signal to interference. Each crystal is supported at its center by four wires; therefore only odd modes, which have a nodal surface at the center, are generated.

The resonant frequency ν_3 of the specimen is calculated from the masses m_1, m_2, m_3 and radii r_1, r_2, r_3 of each component, the resonant frequency of the composite oscillator ν_T, and the resonant frequencies of the quartz crystals ν_1 and ν_2 [36]:

$$\nu_T^2 \sum_{i=1}^{3} m_i r_i^2 = \sum_{i=1}^{3} \nu_i^2 m_i r_i^2$$

The mechanical quality factor Q_T of the composite oscillator is given by

$$Q_T{}^{-1} = 1/\pi \nu_T t$$

in which t is the time required for the vibrations to free-decay to an amplitude $1/e$ times the initial amplitude. The quality factor Q_3 of the specimen is obtained from

$$Q_T{}^{-1} = \sum_{i=1}^{3} \nu_i m_i r_i^2 Q_i{}^{-1} / \nu_T \sum_{i=1}^{3} m_i r_i^2 + Q_g{}^{-1}$$

in which Q_g of the gage circuit is very large and Q_1 and Q_2 are the quality factors of the quartz elements. In the present apparatus, $Q_g{}^{-1}$ is negligible. If the specimen quality factor is much greater than one, we have

$$Q_3{}^{-1} \cong \tan\delta$$

for the specimen. The peak strain, determined from the voltage output of the sensor crystal and the geometry of the plated electrodes, is limited from above by 10^{-3}, the strain at which the quartz crystals fracture, and from below by 10^{-13}, at which thermal fluctuations exceed the signal. Actual strain levels were 10^{-8} to 10^{-7}, well within the linear range for bone.

The elastic compliance of the cement used to assemble the oscillator exceeds that of the quartz and the specimen. Errors due to cement compliance are avoided by using identical quartz crystals and choosing the specimen length so that the specimen resonant frequency is very close to that of the quartz crystals (32.7kHz). Specimen lengths were 27 to 32 mm.

The composite oscillator was tested by examining the resonance of the quartz crystals alone and by examining resonance with a cylinder of polymethyl methacrylate (PMMA). For quartz, a measured shear modulus of 58 GN/m^2 and Q^{-1} = 4 to 7 $\times$ 10^{-5} are consistent with the literature [36]. The mechanical dissipation depends upon the nature of the nodal support, which cannot occur at a mathematical point. If the wires are bent, the dissipation changes, which necessitates repetition of these measurements prior to testing of an unknown. For PMMA, we obtain G = 2 GN/m^2 and tan δ = 0.03 at 32kHz, values which are comparable to Koppelmann's [37] data at 3kHz, G = 1.85 GN/m^2, tan δ = 0.04. The difference is what one might expect from dispersion between 3kHz and 32kHz.

Bone specimens were obtained from fresh frozen donated cadaver tissue. Prismatic sections were rough-cut with a bandsaw from the cortices of long bones and cut slowly on a precision lathe, into a cylindrical shape. Specimens were kept wet with Ringer's solution during cutting and were refrigerated under Ringer's solution between tests. During testing, specimens were kept moist with Ringer's solution by means of a cotton swab. It proved impractical to fully immerse the specimen during testing since viscous drag on an immersed specimen increased its apparent loss tangent by about 60 percent and decreased its resonant frequency by 0.7 percent. Testing was done at room temperature, 29 ± 1°C. The mass of each specimen was determined using a balance with resolution 0.1 mg. Length and diameter were measured with a dial micrometer. These measurements were done on moist specimens; in the mass measurement, the fluid layer was allowed to evaporate to minimize errors due to the mass of adhering fluid. Density was calculated from the mass and dimensions.

Results

In the lowest mode of torsional vibration, the specimen resonant frequency was found to depend on specimen diameter. Figure 2 shows these results in terms of effective shear modulus versus diameter. Classical elasticity and viscoelasticity predict no dependence of modulus on diameter; this corresponds to l = 0 in the couple-stress theory. Couple-stress theory predicts an increase in resonant frequency and therefore effective stiffness, with a decrease in specimen diameter [9]. A theoretical curve for l = 0.09 mm, also shown in Fig. 2, fits the data. The behavior of human compact bone appears to be consistent with couple-stress theory.

The loss tangent had an average value of 0.0525 and a standard deviation of 0.01243. The loss tangent was insensitive to specimen diameter d: A linear regression analysis yielded tan δ = $0.0580 - 0.0017$ d(mm). The value of the loss measured in this study, as well as the apparent shear modulus, is compared with earlier work in Fig. 3.

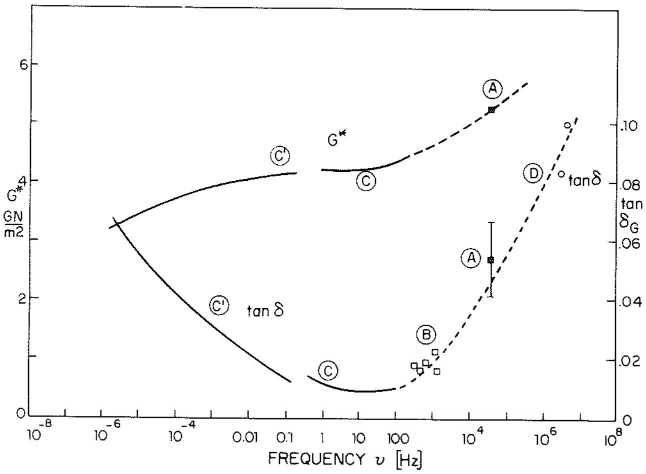

Fig. 3 Loss tangent and effective shear modulus of compact bone—A present study, wet human femoral bone, 29°C; B after Thompson [46], dry whole canine radius; C after Lakes, Katz, and Sternstein [34], wet human tibial bone, 37°C; C′ after Lakes and Katz [47], wet human tibial bone, 37°C (based on a constitutive equation obtained from relaxation data [34]; D after Adler and Cook [48], wet canine bone (from ultrasonic attenuation measurement)

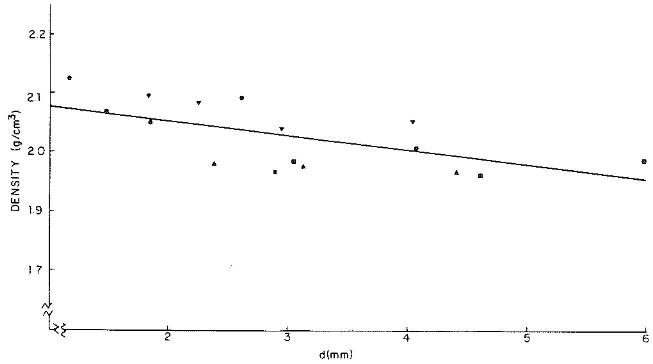

Fig. 4 Density versus specimen diameter. Symbols are as in Fig. 2.

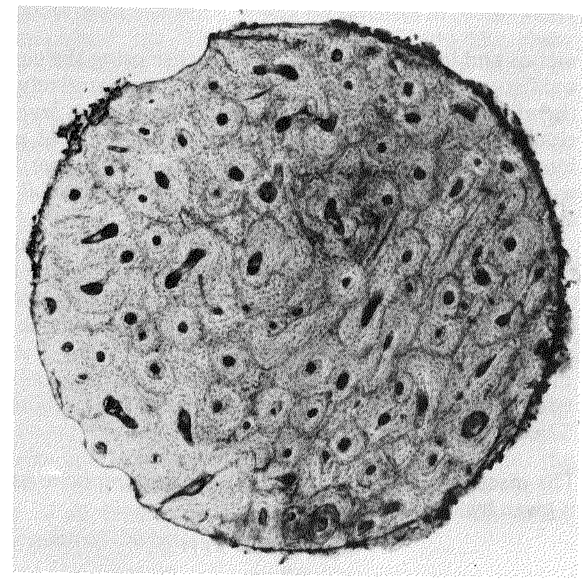

Fig. 5 Transmitted light micrograph of a typical specimen. Specimen diameter is 2.62 mm.

In a later study of fibular bone, a heat lamp was used to vary the specimen temperature. The apparent shear modulus decreased 0.24 percent ± 0.08 percent/°F increase in temperature (0.43 percent/°C). This may be compared with 0.25 percent/°F change in bending compliance of wet human tibial bone observed by Smith and Walmsley [49].

Specimen density exhibits a small dependence upon diameter, as shown in Fig. 4. The range of density variations is 6.4 percent and a linear regression analysis results in $\rho = 2.102 - 0.025\ d(\mathrm{mm})$ (in g/cm^3). The correlation coefficient between density and diameter is $r = 0.572$; therefore about one-third of the fitted density variation is actually associated with specimen size. This may arise from a porosity gradient across the cortex, the mass of adherent Ringer's solution, or from other causes.

Discussion and Conclusion

Torsional vibration studies at 32 kHz disclose size effects in bone which are consistent with couple-stress theory. Let us consider several alternative hypotheses for the observed effects. Size effects may result from a modified constitutive equation, e.g., that of couple-stress theory, or from a dependence of the elastic constants on density, which may in turn depend on size, or from a combination of the two. In a porous material, the Young's modulus E depends on the volume fraction V of pores, as follows [38]: $E = E_o(1 - 1.9V + 0.9V^2)$. In compact bone, a density variation of 6.4 percent should therefore result in a modulus variation of about 12 percent. The graph of modulus versus diameter predicted by the density effect is nearly linear: Substitution of the regression line for the density into the foregoing equation for E/E_o yields

$$\frac{E}{E_o} = 1 - 0.0226d + 1.27\ 10^{-4}d^2$$

in which d is in millimeters. The form of the expected density effect differs from the form of the observed size effect and, in addition, the magnitude of the former is too small (<12 percent) to account for the latter (~ 63 percent). Indeed, the foregoing estimate of the density artifact is probably too high in view of the relatively small correlation r between density and diameter. Since $r^2 = 0.33$, only 0.33 of the density variation is actually associated with specimen size. The modulus variation associated with this is (12 percent)(0.33) = 4 percent.

The range in effective modulus for the data shown in Fig. 2 is 63 percent. In view of these considerations, we attribute most of the size effect, at least 81 percent of it, to couple stress. Some curious size effects might also result from a gradient of mechanical properties across the cortex of a long bone. Yoon and Katz [33] have measured the microhardness of bone and found it to be uniform over the cross section. Other properties, e.g., porosity, may vary over the cross section. Such a variation would perturb the measurement of a large specimen more so than of a small one since a small specimen intercepts a smaller portion of the proposed porosity gradient. The sort of size effect anticipated from a hypothetical porosity gradient is more pronounced for larger diameters than small ones, exactly the opposite of what is observed. The alternative hypotheses are insufficient to account for the observed behavior. We attribute most of it to couple stress.

Couple-stress theory, like classical force stress elasticity theory, is a continuum representation of the phenomenology of mechanical behavior. Neither theory makes explicit reference to microstructure or to deformation mechanisms. Nevertheless, it is possible to derive such theories from microstructural or atomistic models as continuum approximations; generally a higher order of approximation is needed to extract couple stress theory. In derivations of this sort, the characteristic length l tends to be of the order of the size of the structural elements [23–25]. It is perhaps not surprising that the measured characteristic length for bone, $l = 0.09$ mm, is of the order of the size of osteons, which are about 0.15 to 0.25 mm in dia. Typical specimen microstructure is illustrated in Fig. 5. This micrograph is a cross-sectional view of the Haversian architecture of the specimen. The faint dark outlines of a nearly circular shape are cement lines which form the boundaries of osteons. The dark dot at the center of each osteon is a Haversian canal.

The distinction between the phenomenological view and the mechanistic view is a significant one. Some interaction between these views has appeared recently in relation to electromechanical effects in bone. Williams, et al. [39], suggested, on the basis of bending experiments, that the piezoelectric constitutive equation for bone should contain an additional term involving the strain gradient [39]. The model has been criticized as ad hoc [40]. The present author pointed out that the gradient model can be extracted from a more general nonlocal theory and can be related to microstructure [41]. It has recently been reported that the results which led to the gradient-effect model can be accounted for by variations in piezoelectric properties on the scale of osteons. A model which deals with osteons is on the mechanism level, whereas one which contains strains and strain gradients is phenomenological. The two, if correct, should complement rather than conflict with each other. A similar situation can be expected to occur in the case of couple-stress elasticity, which is phenomenological.

The present results indicate that bone has a characteristic length of 0.09 mm at 32kHz, a value which is less than the characteristic length l = 0.15 mm, reported in quasi-static tests for which the effective frequency is 0.5 Hz, on wet bone [42]. This suggests that bone might exhibit "Cosserat viscoelasticity," i.e., a time or frequency dependence of the couple-stress elastic coefficients. Explanation of such behavior in terms of mechanisms is likely to involve the time dependence of the ground substance between osteons. At ultrasonic frequencies ($\sim$1MHz), a model in which the ground substance is one quarter as stiff as the osteon predicts anisotropic elastic behavior similar to what is observed [43]. At low frequencies associated with slow quasi-static tests ($\sim 10^{-7}$Hz), the ground substance at the cement line appears to behave in a viscouslike fashion [44]. In a model of bone as an array of parallel fibers (the osteons) embedded in a compliant matrix (the ground substance), couple-stress effects are predicted [45]. If the matrix becomes very compliant, the predicted couple stress effects become larger. A time-dependent characteristic length, therefore, is an expected consequence of relaxation of the ground substance.

The results reported here are consistent with couple-stress theory but are not consistent with classical (force-stress) elasticity. The results, however, do not discriminate among extended continuum theories, e.g., couple-stress theory, micropolar theory, nonlocal theory. Bone may exhibit degrees of freedom not explored in this study. The examination of these is a subject for future experimentation.

Significance

Couple-stress effects in compact bone have sufficient magnitude to significantly affect the response of specimens a few millimeters or less in diameter. It is unlikely that couple stress will greatly perturb the overall stress distributions in the diaphysis of human long bone. The effects are, however, large enough to modify the stress concentration factors for holes drilled in bone during surgical procedures. It is premature to make quantitative predictions of these stress concentrations, since human compact bone is anisotropic. Although anisotropic constitutive equations have been published, the only solutions available for stress concentration problems appear to be for the isotropic case. Couple stress is also predicted to have a major effect in the vicinity of interfaces. It is in this interfacial region, between bone and an implant material, that problems may develop in the application of skeletal prostheses. Further consideration of the effects of couple stress in bone is therefore warranted.

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